

COMPLETE INTERSECTION CALABI–YAU MANIFOLDS WITH RESPECT TO HOMOGENEOUS VECTOR BUNDLES ON GRASSMANNIANS

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ABSTRACT. Based on the method by [Küc95], we give a procedure to list up all complete intersection Calabi–Yau manifolds with respect to homogeneous vector bundles on Grassmannians for each dimension. In particular, we give a classification of such Calabi–Yau 3-folds and determine their topological invariants. We also give another descriptions for some of them.

1. INTRODUCTION

In this paper, a smooth projective manifold is called *Calabi–Yau* if its first Chern class is trivial. Calabi–Yau manifolds receive significant interest from mathematicians and physicists, not only for the classification of algebraic varieties but also for the relation with the string theory. In dimensions greater than two, there are many different deformation equivalent classes of Calabi–Yau manifolds. Many construction of Calabi–Yau manifolds are known, for instance, complete intersections of hypersurfaces in toric Fano varieties (see e.g. [CK99] and references therein). However, it is still an open problem whether or not the number of deformation equivalent classes of Calabi–Yau 3-folds is finite.

Let \mathcal{F} be a globally generated homogeneous vector bundle on the Grassmannian $G(k, n)$ of k dimensional subspaces in \mathbb{C}^n . We denote by $Z_{\mathcal{F}} \subset G(k, n)$ the zero locus of a general global section of \mathcal{F} . From the Bertini type theorem by [Muk92], $Z_{\mathcal{F}}$ is a disjoint union of smooth submanifolds of $G(k, n)$ with $\text{codim } Z_{\mathcal{F}} = \text{rank } \mathcal{F}$ if it is not the empty set. If that is the case, we call $Z_{\mathcal{F}}$ a *complete intersection* with respect to \mathcal{F} .

Küchle has classified complete intersection Fano 4-folds with respect to homogeneous vector bundles on Grassmannians [Küc95]. In this paper, we slightly generalize his result and obtain the list of all complete intersection Calabi–Yau 3-folds with respect to homogeneous vector bundles on Grassmannians.

Theorem 1.1. *Let $1 < k < n - 1$ be integers. A complete intersection Calabi–Yau 3-fold $Z_{\mathcal{F}}$ in $G(k, n)$ for a globally generated homogeneous vector bundle \mathcal{F} is one of the listed 3-folds in Table 1 up to natural identifications among the Grassmannians explained in Remark 3.1. All except for No. 30 are irreducible, and all except for No. 26 are Calabi–Yau 3-folds in the strict sense, i.e. $h^1(\mathcal{O}_{Z_{\mathcal{F}}}) = h^2(\mathcal{O}_{Z_{\mathcal{F}}}) = 0$.*

In [Kuz15], Kuznetsov gave alternative descriptions of the Fano 4-folds in [Küc95] with Picard number greater than 1, i.e. (b4), (b9), (c7), (d3) in Küchle’s list. In the paper, Kuznetsov asked whether $Z_{\mathcal{Q}(1)} \subset G(2, 6)$ and $Z_{\mathcal{Q}(1) \oplus 6} \subset G(2, 7)$ are deformation equivalent or not since they have the same collections of discrete invariants, where \mathcal{Q} is the universal quotient bundle of rank 4 on $G(2, 6)$. Soon after a preliminary version of [Kuz15] appeared, Manivel gave an affirmative answer in [Man15]. In fact, Manivel showed that these two types of varieties are the same up to projective

equivalence. In Sections 4 to 7, we also give alternative descriptions of most of Calabi–Yau 3-folds in Table 1.

No.	$G(k, n)$	\mathcal{F}	H^3	$c_2 \cdot H$	c_3	$h^{1,1}$	[Küc95]	Description
1	$G(2, 4)$	$\mathcal{O}(4)$	8	56	−176	1		$(\mathbb{P}^5)_{2,4}$
2	$G(2, 5)$	$\mathcal{O}(1) \oplus \mathcal{O}(2)^{\oplus 2}$	20	68	−120	1	(b2)	$G(2, 5)_{1,2^2}$
3		$\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(3)$	15	66	−150	1	(b1)	$G(2, 5)_{1^2,3}$
4		$\mathcal{S}^*(1) \oplus \mathcal{O}(2)$	24	72	−116	1		Remark 4.4
5		$\wedge^2 \mathcal{Q}(1)$	25	70	−100	1		Proposition 4.7
6	$G(2, 6)$	$\mathcal{O}(1)^{\oplus 4} \oplus \mathcal{O}(2)$	28	76	−116	1	(b6)	$G(2, 6)_{1^4,2}$
7		$\mathcal{S}^*(1) \oplus \mathcal{O}(1)^{\oplus 3}$	33	78	−102	1	(b5), I	Remark 4.4
8		$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	40	88	−128	2	(b4)	$(\mathbb{P}^3 \times \mathbb{P}^3)_{1^2,2}$ [Kuz15]
9		$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{S}^*(1)$	48	84	−92	2		
10		$\mathcal{Q}(1) \oplus \mathcal{O}(1)$	42	84	−98	1	(b3), II	[Man15]
11		$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(3)$	18	72	−162	2		$(\mathbb{P}^2 \times \mathbb{P}^2)_3$
12	$G(2, 7)$	$\mathcal{O}(1)^{\oplus 7}$	42	84	−98	1	(b7), III	$G(2, 7)_{1^7}$
13		$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 4}$	56	92	−92	1	(b8), V	$OG(2, 7)_{1^4}$
14		$(\mathrm{Sym}^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)$	80	80	−32	8	(b9)	[Cas15]
15		$\wedge^4 \mathcal{Q} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$	36	84	−120	1	(b10)	$(G_2/P_1)_{1,2}$
16		$\mathcal{S}^*(1) \oplus \wedge^4 \mathcal{Q}$	42	84	−98	1		Proposition 5.1
17	$G(2, 8)$	$\wedge^5 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 3}$	57	90	−84	1	(b11), VI	Proposition 6.1
18		$\mathrm{Sym}^2 \mathcal{S}^* \oplus \wedge^5 \mathcal{Q}$	72	96	−72	1		Proposition 6.1
19	$G(3, 6)$	$\mathcal{O}(1)^{\oplus 6}$	42	84	−96	1	(c1), IV	$G(3, 6)_{1^6}$
20		$\wedge^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$	32	80	−116	1	(c2)	$LG(3, 6)_{1^2,2}$
21		$\mathcal{S}^*(1) \oplus \wedge^2 \mathcal{S}^*$	42	84	−96	1		Proposition 5.1
22	$G(3, 7)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}$	128	128	−128	1	(c4)	$(\mathbb{P}^7)_{2^4}$
23		$(\wedge^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 3}$	61	94	−86	1	(c6), VII	
24		$(\wedge^3 \mathcal{Q})^{\oplus 2} \oplus \mathcal{O}(1)$	72	96	−74	1	(c3), IX	
25		$\wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$	66	96	−84	1	(c5), VIII	[Kuz16]
26	$G(3, 8)$	$(\mathrm{Sym}^2 \mathcal{S}^*)^{\oplus 2}$	384	0	0	9		[Rei72], Theorem 7.1
27		$\mathrm{Sym}^2 \mathcal{S}^* \oplus (\wedge^2 \mathcal{S}^*)^{\oplus 2}$	176	128	−64	2		
28		$(\wedge^2 \mathcal{S}^*)^{\oplus 4}$	92	104	−64	1		
29		$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$	102	108	−84	2	(c7)	[Kuz15]
30	$G(4, 8)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}$	256	256	−256	2	(d1)	$(\mathbb{P}^7)_{2^4} \sqcup (\mathbb{P}^7)_{2^4}$
31		$(\wedge^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(2)$	48	96	−128	4		$(\Pi^4 \mathbb{P}^1)_2$ [Kuz15]
32	$G(4, 9)$	$\mathrm{Sym}^2 \mathcal{S}^* \oplus \wedge^2 \mathcal{S}^* \oplus \mathcal{O}(1)$	384	192	−128	4	(d2)	$(\Pi^4 \mathbb{P}^1)_2$
33	$G(5, 10)$	$(\wedge^2 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$	120	180	−220	5	(d3)	$(\Pi^5 \mathbb{P}^1)_{1^2}$ [Kuz15]

TABLE 1. complete intersection Calabi–Yau 3-folds in Grassmannians

In the right-most column of Table 1, we use the notation X_{d_1, \dots, d_r} , which means a complete intersection of r general hypersurfaces of degree d_1, \dots, d_r in X . If X is one of Grassmannians $G(k, n)$, orthogonal Grassmannians $OG(k, n)$, Lagrangian Grassmannians $LG(k, 2k)$ and a G_2 -Grassmannian G_2/P_1 , the degrees are defined with respect to the unique ample generator of the

Picard group. For the case of $X = \prod^l \mathbb{P}^s$, the degrees of the hypersurfaces appearing here are defined with respect to $\mathcal{O}_X(1, \dots, 1)$.

This paper is organized as follows. In Section 2, we show Proposition 2.2, which states that there are at most finitely many families of complete intersection Calabi–Yau d -folds with respect to homogeneous vector bundles on Grassmannians for a fixed dimension $d > 0$. The proof gives a concrete procedure to classify such d -folds. In Section 3, we exemplify the classification of Calabi–Yau 3-folds, and show Theorem 1.1. In Sections 4 to 7, we study alternative descriptions of Calabi–Yau 3-folds in Table 1. Throughout this paper, we work over the complex number field \mathbb{C} .

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2. FINITENESS OF COMPLETE INTERSECTION CALABI–YAU MANIFOLDS

In this section, we give a procedure to list up all homogeneous vector bundles \mathcal{F} on Grassmannians such that $Z_{\mathcal{F}}$ is a d -dimensional Calabi–Yau manifolds for fixed $d > 0$.

Let \mathcal{S} and \mathcal{Q} be the universal subbundle and the universal quotient bundle on a Grassmannian $G(k, n)$, respectively. Hence there exists an exact sequence on $G(k, n)$

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{Q} \rightarrow 0. \quad (1)$$

We denote by $S_{\lambda}\mathcal{S}^*$ (resp. $S_{\mu}\mathcal{Q}$) the globally generated homogeneous vector bundle corresponding to the Schur module $S_{\lambda}\mathbb{C}^k$ (resp. $S_{\mu}\mathbb{C}^{n-k}$) with respect to a Young diagram $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$ of $GL(k)$ (resp. $\mu = (\mu_1 \geq \dots \geq \mu_{n-k} \geq 0)$ of $GL(n-k)$).

Any irreducible globally generated homogeneous vector bundle \mathcal{E} on $G(k, n)$ can be written as

$$\mathcal{E} = S_{\lambda}\mathcal{S}^* \otimes S_{\mu}\mathcal{Q} \otimes \mathcal{O}(p) \quad (2)$$

for some λ, μ and $p \geq 0$ with $\lambda_k = 0$ and $\mu_{n-k} = 0$. We note that $S_{\lambda}\mathcal{S}^*(p) = S_{\lambda}\mathcal{S}^* \otimes \mathcal{O}(p)$ is isomorphic to $S_{\lambda'}\mathcal{S}^*$ for $\lambda' = (\lambda'_1 \geq \dots \geq \lambda'_k \geq 0)$ with $\lambda'_j = \lambda_j + p$.

Let \mathcal{F} be a globally generated homogeneous vector bundle on a Grassmannian $G(k, n)$ such that $\dim Z_{\mathcal{F}} = d$ and $c_1(Z_{\mathcal{F}}) = 0$. By the adjunction formula for $Z_{\mathcal{F}} \subset G(k, n)$, it holds that

$$\text{rank } \mathcal{F} = k(n-k) - d \quad \text{and} \quad c_1(\mathcal{F}) = n. \quad (3)$$

The condition (3) imposes several restrictions on an irreducible component $\mathcal{E} \subset \mathcal{F}$. One of the restrictions is the following.

Lemma 2.1 ([Küc95, Corollary 3.5 (a)]). *Let \mathcal{E} be an irreducible component of \mathcal{F} . Then \mathcal{E} is one of $S_{\lambda}\mathcal{S}^*$, $S_{\mu}\mathcal{Q}$, and $\mathcal{O}(p)$ for $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$ with $\lambda_1 \neq \lambda_k$, $\mu = (\mu_1 \geq \dots \geq \mu_{n-k} \geq 0)$ with $\mu_1 \neq \mu_{n-k}$ and $p > 0$, respectively.*

Proof. Otherwise, it contradicts $d > 0$ since $\text{rank } \mathcal{F} \geq \text{rank } \mathcal{E} \geq k(n-k)$. □

In this section, we show the following proposition.

Proposition 2.2. *Consider a globally generated homogeneous vector bundle \mathcal{F} on $G(k, n)$ which is decomposed as*

$$\mathcal{F} = \mathcal{F}^{\mathcal{S}} \oplus \mathcal{F}^{\mathcal{Q}} \oplus \mathcal{F}^{\text{line}}, \quad (4)$$

where $\mathcal{F}^{\mathcal{S}}$, $\mathcal{F}^{\mathcal{Q}}$ and $\mathcal{F}^{\text{line}}$ are direct sums of irreducible homogeneous vector bundles of type $S_{\lambda}\mathcal{S}^*$, $S_{\mu}\mathcal{Q}$, and $\mathcal{O}(p)$ with $\lambda_1 \neq \lambda_k \geq 0$, $\mu_1 \neq \mu_{n-k} \geq 0$, and $p > 0$, respectively.

For a fixed positive integer $d > 0$, there are at most finitely many choice of positive integers k, n , and \mathcal{F} on $G(k, n)$ such that (3) and the following assumptions (A1), ..., (A4) are satisfied.

(A1) $k \geq 2$,

(A2) $n \geq 2k$,

(A3) $\mathcal{S}^* \not\subset \mathcal{F}^{\mathcal{S}}$ and $\mathcal{Q} \not\subset \mathcal{F}^{\mathcal{Q}}$,

(A4) $\text{Sym}^2 \mathcal{Q} \not\subset \mathcal{F}^{\mathcal{Q}}$ and $\wedge^2 \mathcal{Q} \not\subset \mathcal{F}^{\mathcal{Q}}$ for $n > 2k$,

where $\mathcal{E} \not\subset \mathcal{E}'$ means that \mathcal{E} is not contained in \mathcal{E}' as an irreducible component.

Here we explain each assumption. Let \mathcal{F} be a globally generated homogeneous vector bundle on $G(k, n)$ such that $Z_{\mathcal{F}}$ is a Calabi–Yau d -fold.

By Lemma 2.1, such \mathcal{F} must be decomposed as (4).

If $k = 1$, i.e. if \mathcal{F} is on a projective space \mathbb{P}^{n-1} , we easily see that $\mathcal{F}^{\mathcal{S}} = \mathcal{F}^{\mathcal{Q}} = 0$ because $\mathcal{S}^* = \mathcal{O}(1)$, $\text{rank } \mathcal{Q} = n - 1$, and $d > 0$. Hence we may assume (A1) unless we consider complete intersections of hypersurfaces in projective spaces.

Set $l = n - k$. Under the natural isomorphism $G(k, n) \simeq G(l, n)$, the homogeneous vector bundles $S_{\lambda}\mathcal{S}^*$ and $S_{\mu}\mathcal{Q}$ on $G(k, n)$ are transformed to $S_{\lambda}\mathcal{Q}$ and $S_{\mu}\mathcal{S}^*$ on $G(l, n)$, respectively. Hence we assume (A2) to avoid the duplication.

There is another natural isomorphism between $Z_{\mathcal{S}^*} \subset G(k, n)$ and $G(k, n - 1)$. Under the isomorphism, the restrictions of homogeneous vector bundles \mathcal{S}^* and \mathcal{Q} on $Z_{\mathcal{S}^*}$ correspond to \mathcal{S}^* and $\mathcal{Q} \oplus \mathcal{O}$ on $G(k, n - 1)$, respectively. A similar property holds for $Z_{\mathcal{Q}} \subset G(k, n)$ and $G(k - 1, n - 1)$. Hence we assume (A3) as in [Küc95, Lemma 3.2 (ii)].

Finally, by considering the possible dimension of an isotropic subspace in \mathbb{C}^n equipped with a symmetric or a skew-symmetric form of the maximal rank, we see that $Z_{\text{Sym}^2 \mathcal{Q}} = \emptyset$ for $n > 2k$ and $Z_{\wedge^2 \mathcal{Q}} = \emptyset$ for $n > 2k + 1$. Furthermore, $Z_{\wedge^2 \mathcal{Q}} \subset G(k, 2k + 1)$ is isomorphic to $Z_{\wedge^2 \mathcal{Q}} \subset G(k, 2k)$, under which the restrictions of homogeneous vector bundles on $Z_{\wedge^2 \mathcal{Q}} \subset G(k, 2k + 1)$ are transformed to other kinds of homogeneous vector bundles on $Z_{\wedge^2 \mathcal{Q}} \subset G(k, 2k)$. Therefore we also assume $\wedge^2 \mathcal{Q} \not\subset \mathcal{F}$ for $n = 2k + 1$. Hence we assume (A4) as in [Küc95, Lemma 3.2 (iii)].

In conclusion, we have the following corollary of Proposition 2.2.

Corollary 2.3. *For a fixed positive integer $d > 0$, there are at most finitely many families of complete intersection Calabi–Yau d -folds in Grassmannians with respect to globally generated homogeneous vector bundles, up to natural identifications among Grassmannians explained as above.*

In the rest of this section, we give a proof of Proposition 2.2. To show Proposition 2.2, we may assume one more condition for $n = 2k$.

Lemma 2.4. *To prove Proposition 2.2, it suffices to show the finiteness of the choices of k, n , and \mathcal{F} under the additional condition*

(A5) $\mathcal{F}^{\mathcal{Q}} = 0$ if $n = 2k$.

Proof. Assume $n = 2k$ and take $\mathcal{F} = \mathcal{F}^{\mathcal{S}} \oplus \mathcal{F}^{\mathcal{Q}} \oplus \mathcal{F}^{\text{line}}$ which satisfies (3) and (A1), ..., (A4). Write $\mathcal{F}^{\mathcal{Q}} = \bigoplus_{\mu} (S_{\mu}\mathcal{Q})^{\oplus a_{\mu}}$ for $a_{\mu} \geq 0$. Since $\text{rank } \mathcal{S}^* = \text{rank } \mathcal{Q}$ and $c_1(\mathcal{S}^*) = c_1(\mathcal{Q})$, $\mathcal{F}' := \mathcal{F}'^{\mathcal{S}} \oplus \mathcal{F}^{\text{line}}$ also satisfies (3) and (A1), ..., (A4) for $\mathcal{F}'^{\mathcal{S}} := \mathcal{F}^{\mathcal{S}} \oplus \bigoplus_{\mu} (S_{\mu}\mathcal{S}^*)^{\oplus a_{\mu}}$.

Then \mathcal{F} is obtained from \mathcal{F}' by replacing some components $S_\lambda \mathcal{S}^*$ of \mathcal{F}' by $S_\lambda \mathcal{Q}$. Since we have at most finitely many \mathcal{F} by such replacement for a fixed \mathcal{F}' , it suffices to show the finiteness of \mathcal{F}' , which satisfies the additional condition (A5). \square

Example 2.5. Consider $\mathcal{F}_1 = (\wedge^3 \mathcal{S}^*)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$ on $G(4, 8)$. Then $\text{rank } \mathcal{F}_1 = 10$, $c_1(\mathcal{F}_1) = 8$, and hence $Z_{\mathcal{F}_1}$ is a Calabi–Yau 6-fold. Replacing one of the irreducible component $\wedge^3 \mathcal{S}^* \subset \mathcal{F}$ by $\wedge^3 \mathcal{Q}$, we obtain $\mathcal{F}_2 = \wedge^3 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$, from which we have another Calabi–Yau 6-fold $Z_{\mathcal{F}_2}$. Since the Euler numbers of these varieties are computed as $\chi(Z_{\mathcal{F}_1}) = 14148$ and $\chi(Z_{\mathcal{F}_2}) = 14328$, we see that $Z_{\mathcal{F}_1}$ and $Z_{\mathcal{F}_2}$ are not isomorphic.

Throughout Subsections 2.1, 2.2, $\mathcal{F} = \mathcal{F}^{\mathcal{S}} \oplus \mathcal{F}^{\mathcal{Q}} \oplus \mathcal{F}^{\text{line}}$ is a vector bundle as in Proposition 2.2, which satisfies (2) and (A1)-(A5).

2.1. The case with $\mathcal{F}^{\mathcal{Q}} \neq (\wedge^{l-1} \mathcal{Q})^{\oplus A} \oplus (\mathcal{Q}(1))^{\oplus B}$. We begin with the case $\mathcal{F}^{\mathcal{Q}} \neq 0$. In this case, we have $n > 2k$ by (A2) and (A5). Let us write $l := n - k > k$.

Lemma 2.6. *Assume that there exists an irreducible component \mathcal{E} of $\mathcal{F}^{\mathcal{Q}}$ which is not $\wedge^{l-1} \mathcal{Q}$ nor $\mathcal{Q}(1)$. Then \mathcal{F} is one of the following:*

	$G(k, n)$	\mathcal{F}	$d = \dim Z_{\mathcal{F}}$
$(\alpha 1)$	$G(3, 8)$	$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(2)$	4
$(\alpha 2)$		$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 2}$	3
$(\alpha 3)$		$\wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q}$	2
$(\alpha 4)$	$G(4, 9)$	$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(3)$	9
$(\alpha 5)$		$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$	8
$(\alpha 6)$		$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)^{\oplus 3}$	7
$(\alpha 7)$		$\wedge^2 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q}$	4
$(\alpha 8)$		$\wedge^3 \mathcal{S}^* \oplus \wedge^3 \mathcal{Q}$	6
$(\alpha 9)$	$G(4, 10)$	$\wedge^3 \mathcal{Q}$	4
$(\alpha 10)$		$\wedge^4 \mathcal{Q}$	9
$(\alpha 11)$	$G(5, 11)$	$\wedge^3 \mathcal{Q} \oplus \mathcal{O}(1)$	9
$(\alpha 12)$		$\wedge^4 \mathcal{Q} \oplus \mathcal{O}(1)$	14
(βk)	$G(k, 2k + 1)$	$\wedge^k \mathcal{Q}(1)$	$k^2 - 1$

Proof. Suppose that $\mathcal{F}^{\mathcal{Q}}$ has an irreducible component $\mathcal{E} = S_\mu \mathcal{Q} \neq \wedge^{l-1} \mathcal{Q}$ nor $\mathcal{Q}(1)$, i.e. $\mu \neq (1, \dots, 1, 0)$ nor $(2, 1, \dots, 1)$. There are the following possible cases for μ ,

- (Q1) $\mu_i > \mu_{i+1}$ and $\mu_j > \mu_{j+1}$ for some $i \neq j$,
- (Q2) $\mu_i - \mu_{i+1} \geq 2$ for some $2 \leq i \leq l - 2$, and $l \geq 4$,
- (Q3) $\mu_{l-1} - \mu_l \geq 2$,
- (Q4) $\mu_1 - \mu_2 \geq 2$,
- (Q5) $\mu_i - \mu_{i+1} = 1$ for some $3 \leq i \leq l - 3$, and $l \geq 6$,
- (Q6) $\mu_{l-2} - \mu_{l-1} = 1$ and $l \geq 5$,
- (Q7) $\mu_i - \mu_{i+1} = 1$ for $i = 2$ or $l - 1$,

$$(Q8) \quad \mu_1 - \mu_2 = 1,$$

where we suppose the i -th case (Q_i) does not include any μ which satisfies one of the conditions from the first case $(Q1)$ to the $(i-1)$ -th case $(Q(i-1))$ for all $i = 2, \dots, 8$.

We can list up all $\mathcal{E} = S_\mu \mathcal{Q}$ which satisfy the conditions $\delta := kl - \text{rank } \mathcal{E} = \dim Z_{\mathcal{E}} > 0$ and $\iota := n - c_1(\mathcal{E}) = c_1(Z_{\mathcal{E}}) \geq 0$ for each case by using the formulas

$$\text{rank } \mathcal{E} = \prod_{1 \leq i < j \leq l} \frac{j - i + \mu_i - \mu_j}{j - i}, \quad (5)$$

$$c_1(\mathcal{E}) = \frac{|\mu|}{l} \text{rank } \mathcal{E}, \quad (6)$$

where $|\mu| = \mu_1 + \dots + \mu_l$.

First, we omit the case $(Q1)$. If that is the case,

$$\text{rank } S_\mu \mathcal{Q} \geq \text{rank } S_{(2^i, 1^{j-i}, 0^{l-j})} \mathcal{Q} \geq \text{rank } S_{(2, 1^{j-1}, 0^{l-j})} \mathcal{Q} \quad (7)$$

$$\geq \text{rank } S_{(2, 1^{l-2}, 0)} \mathcal{Q} = l^2 - 1. \quad (8)$$

Then we have $\delta \leq kl - (l^2 - 1) < 0$ (since $l > k$). This contradicts the condition $\delta > 0$.

We can also exclude $(Q2)$, $(Q3)$ and $(Q4)$. First, let us consider the case $(Q2)$. In this case, we have

$$\text{rank } S_\mu \mathcal{Q} \geq \text{rank } S_{(2^i, 0^{l-i})} \mathcal{Q} \geq \text{rank } S_{(2^2, 0^{l-2})} \mathcal{Q} = \frac{1}{12} l^2 (l^2 - 1) \quad (9)$$

for all $2 \leq i \leq l-2$. Then we obtain $\delta \leq kl - \frac{1}{12} l^2 (l^2 - 1) \leq 0$ (since $l \geq 4$), which is a contradiction.

For $(Q3)$ and $(Q4)$, we use the condition $\iota \geq 0$. Let us treat with the case $(Q3)$ first, i.e. $\mathcal{E} = S_{(j^{l-1}, 0)} \mathcal{Q}(p)$ with $j \geq 2$ and $p \geq 0$. It holds that

$$c_1(\mathcal{E}) = \binom{l+j-1}{j} \left(\frac{j(l-1)}{l} + p \right) \geq \binom{l+1}{2} \left(\frac{2(l-1)}{l} \right) = l^2 - 1. \quad (10)$$

Hence we have a contradiction $\iota \leq n - (l^2 - 1) < 0$ (since $n < 2l$ and $l \geq 3$).

Consider the case $(Q4)$, i.e. $\mathcal{E} = S_{(j, 0^{l-1})} \mathcal{Q}(p) = \text{Sym}^j \mathcal{Q}(p)$ with $j \geq 2$ and $p \geq 0$. Note that the case with $j = 2$ and $p = 0$ is already excluded by the assumption $(A4)$. Then we have

$$c_1(\mathcal{E}) = \binom{l+j-1}{j} \left(\frac{j}{l} + p \right) \geq \min \left\{ \binom{l+1}{2} \left(\frac{2}{l} + 1 \right), \binom{l+2}{3} \frac{3}{l} \right\} = \frac{1}{2} (l+1)(l+2). \quad (11)$$

It also leads us to $\iota \leq n - \frac{1}{2} (l+1)(l+2) < 0$.

The case $(Q5)$ is the first case which gives a non-trivial contribution to the classification. From $3 \leq i \leq l-3$, it holds that $\text{rank } \mathcal{E} \geq \binom{l}{3}$. Hence we have

$$0 < \delta = kl - \text{rank } \mathcal{E} \leq \frac{l}{6} (6k - (l-1)(l-2)) \leq \frac{kl}{6} (7-k), \quad (12)$$

where the last inequality follows from $l \geq k+1$. These inequalities give $(l-1)(l-2) < 6k$ and $k < 7$. Hence all the possible irreducible bundles \mathcal{E} of $(Q5)$ with $\delta > 0$ can be listed in the

following.

\mathcal{E}	$G(k, n)$	$\delta = \dim Z_{\mathcal{E}}$	$\iota = c_1(Z_{\mathcal{E}})$
$\wedge^3 \mathcal{Q}(p)$	$G(4, 10)$	4	$-20p$
	$G(5, 11)$	10	$1 - 20p$
	$G(6, 13)$	7	$-2 - 35p$
$\wedge^4 \mathcal{Q}(p)$	$G(6, 13)$	7	$-7 - 35p$

The condition $\iota \geq 0$ gives two examples of \mathcal{F} , $(\alpha 9)$ and $(\alpha 11)$ in this case (Q5).

In the case (Q6), the condition of μ turns into $\mathcal{E} = \wedge^{l-2} \mathcal{Q}(p)$ with $p \geq 0$. If $p \geq 1$, the condition $\iota = n - c_1(\mathcal{E}) \geq 0$ gives a contradiction,

$$n \geq c_1(\mathcal{E}) = \binom{l-1}{2} + p \binom{l}{2} \geq (l-1)^2 > n \quad (13)$$

since $n \leq 2l - 1$ and $l \geq 5$. Let $p = 0$. We have a condition

$$0 \leq \iota = n - \binom{l-1}{2} \leq \frac{1}{2}(-l^2 + 7l - 4) \quad (\text{since } n \leq 2l - 1). \quad (14)$$

This gives a bound $l \leq 6$, i.e. $l = 5$ or 6 . Together with $l \geq k + 1$, one can list all possible (k, l) by using another condition

$$0 < \delta = kl - \text{rank } \mathcal{E} = kl - \binom{l}{2}. \quad (15)$$

Namely, the possible irreducible vector bundle $\mathcal{E} = \wedge^{l-2} \mathcal{Q}$ of (Q6) with $\delta > 0$ and $\iota \geq 0$ should be over one of the following Grassmannians. The values δ and ι are also listed.

$G(k, n)$	$G(3, 8)$	$G(4, 9)$	$G(4, 10)$	$G(5, 11)$
δ	5	10	9	15
ι	2	3	0	1

We note that $\iota \leq k - 1$ holds for all cases. Hence we have

$$c_1(\mathcal{E}') \leq n - c_1(\mathcal{E}) = \iota \leq k - 1 \quad (16)$$

for any irreducible component \mathcal{E}' of \mathcal{F} other than \mathcal{E} . Thus such \mathcal{E}' is $\mathcal{O}(p)$ for some $0 < p \leq k - 1$, $\wedge^2 \mathcal{S}^*$ or $\wedge^{k-1} \mathcal{S}^*$. Hence we obtain the remaining ten Calabi–Yau manifolds among $(\alpha 1)$ – $(\alpha 12)$.

For (Q7), we may assume $\mathcal{E} = \wedge^2 \mathcal{Q}(p)$ with $p \geq 1$ or $\mathcal{E} = \wedge^{l-1} \mathcal{Q}(p)$ with $p \geq 1$ by the assumptions (A4) and $\mathcal{E} \neq \wedge^{l-1} \mathcal{Q}$, respectively. In the former case, it holds that

$$n \geq c_1(\mathcal{E}) = l - 1 + p \binom{l}{2} \geq l - 1 + \binom{k+1}{2} \geq l + k = n \quad (17)$$

for any $k \geq 2$. The equalities hold only for $k = 2$, $l = k + 1 = 3$ and $p = 1$, i.e. the case $(\beta 2)$. In the latter case, we also observe

$$n \geq c_1(\mathcal{E}) = l - 1 + pl \geq 2l - 1 \geq n, \quad (18)$$

since $p \geq 1$ and $l > \frac{n}{2}$. The both equalities hold simultaneously only for (βk) 's.

Finally, we consider the case (Q8), i.e. $\mathcal{E} = \mathcal{Q}(p)$ with $p \geq 0$. We may assume $p \geq 2$ by the assumptions (A3) and $\mathcal{E} \neq \mathcal{Q}(1)$. Hence we have a contradiction

$$n \geq c_1(\mathcal{E}) = pl + 1 \geq 2l + 1 > n \quad (19)$$

since $p \geq 2$ and $l > \frac{n}{2}$. Therefore the proof is completed. \square

2.2. The case with $\mathcal{F}^{\mathcal{Q}} = (\wedge^{l-1} \mathcal{Q})^{\oplus A} \oplus (\mathcal{Q}(1))^{\oplus B}$. From Lemma 2.6, we may focus on the case

$$\mathcal{F}^{\mathcal{Q}} = (\wedge^{l-1} \mathcal{Q})^{\oplus A} \oplus (\mathcal{Q}(1))^{\oplus B}, \quad (20)$$

where A and B are non-negative integers. Note that the assumptions (A2) and (A5) imply $n \geq 2k+1$ for $(A, B) \neq (0, 0)$ and $n \geq 2k$ for $(A, B) = (0, 0)$. We set $\mathcal{G} := \mathcal{F}^{\mathcal{S}} \oplus \mathcal{F}^{\text{line}}$ for the sake of simplicity.

Lemma 2.7. *The possible values of (A, B) are only $(2, 0), (1, 0), (0, 1)$, and $(0, 0)$.*

Proof. We note that \mathcal{G} must satisfy the conditions

$$\text{rank } \mathcal{G} = kl - (A + B)l - d, \quad (21)$$

$$c_1(\mathcal{G}) = n - A(l-1) - B(l+1). \quad (22)$$

Since \mathcal{G} is globally generated, we have $c_1(\mathcal{G}) \geq 0$. If $A \geq 3$, (22) gives $0 \leq n - 3(l-1) = -2n + 3k + 3$. This contradicts to the assumption (A1) and $n \geq 2k+1$. Similarly, (22) gives $0 \leq -n + 2k - 2$ (resp. $0 \leq -n + 2k$) if $B \geq 2$ (resp. if $A \geq 1$ and $B \geq 1$). They contradict to $n \geq 2k+1$ for $(A, B) \neq (0, 0)$. \square

Lemma 2.8. *If $(A, B) = (2, 0)$, \mathcal{F} is one of the following.*

	$G(k, n)$	\mathcal{F}	$d = \dim Z_{\mathcal{F}}$
(γk)	$G(k, 2k+1)$	$(\wedge^k \mathcal{Q})^{\oplus 2} \oplus \mathcal{O}(1)$	$k^2 - k - 3$
(δk)	$G(k, 2k+2)$	$(\wedge^{k+1} \mathcal{Q})^{\oplus 2}$	$k^2 - 4$

Proof. From (22) and $l \geq k+1$, we see $c_1(\mathcal{G}) = k+2-l = 0$ or 1 . If there exists an irreducible component $\mathcal{E} = S_{\lambda} \mathcal{S}^* \subset \mathcal{F}^{\mathcal{S}}$, we have a contradiction $1 \geq c_1(\mathcal{G}) \geq c_1(\mathcal{E}) = \frac{|\lambda|}{k} \text{rank } \mathcal{E} \geq |\lambda| \geq 2$. Hence there exists no such \mathcal{E} , i.e. $\mathcal{G} = \mathcal{F}^{\text{line}}$. Therefore we also obtain (γk) if $c_1(\mathcal{G}) = 1$ and (δk) if $c_1(\mathcal{G}) = 0$. \square

Now we consider the case with $A + B \leq 1$. Getting the idea from [Küc95], we consider an invariant $\kappa_{\mathcal{E}} := kc_1(\mathcal{E}) - \text{rank } \mathcal{E}$ for a vector bundle \mathcal{E} on $G(k, n)$. This is an additive integral invariant, and takes a positive value

$$\kappa_{\mathcal{E}} = (|\lambda| - 1) \text{rank } \mathcal{E} \quad (23)$$

for any irreducible component $\mathcal{E} = S_{\lambda} \mathcal{S}^* \subset \mathcal{G}$. In particular, we have conditions $\kappa_{\mathcal{E}} \leq \kappa_{\mathcal{G}}$ and

$$\kappa_{\mathcal{G}} = k^2 + d - A(kl - k - l) - B(kl + k - l), \quad (24)$$

from (21) and (22).

In Lemmas 2.6 and 2.8, we describe \mathcal{F} explicitly for all $d > 0$. On the other hand, we only show the finiteness of \mathcal{F} for a fixed $d > 0$ in the following lemmas. Hence we fix a positive integer $d > 0$ in the rest of this section.

Lemma 2.9. *For a fixed k , there are at most finitely many \mathcal{F} with $A + B \leq 1$.*

Proof. We set $a_\lambda \geq 0$ as the multiplicity of any irreducible component $S_\lambda \mathcal{S}^* \subset \mathcal{G}$ with $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$. Note that we allow to choose $\lambda = (p, p, \dots, p)$ with $p > 0$, which corresponds to the line bundle $\mathcal{O}(p) \subset \mathcal{F}^{\text{line}}$. If $A + B = 1$, it holds that

$$\sum_{\lambda} a_{\lambda} c_1(S_{\lambda} \mathcal{S}^*) = c_1(\mathcal{G}) = k \pm 1, \quad (25)$$

from (22). The finiteness of the choices of λ and (a_{λ}) follows from (25) and

$$c_1(S_{\lambda} \mathcal{S}^*) = \frac{|\lambda|}{k} \text{rank } S_{\lambda} \mathcal{S}^* \geq \frac{|\lambda|}{k} > 0. \quad (26)$$

If $A = B = 0$,

$$\sum_{\lambda} a_{\lambda} \kappa_{S_{\lambda} \mathcal{S}^*} = \kappa_{\mathcal{G}} = k^2 + d \quad (27)$$

holds from (24). The finiteness of the choices of λ and (a_{λ}) also follows from

$$\kappa_{S_{\lambda} \mathcal{S}^*} = (|\lambda| - 1) \text{rank } \mathcal{E} \geq |\lambda| - 1 > 0. \quad (28)$$

□

From Lemma 2.9, it suffices to show the boundedness of k for $A + B \leq 1$. Let us define

$$\varphi_{\mathcal{E}}(k) := \kappa_{\mathcal{E}} - k^2 + A(k^2 - k - 1) + B(k^2 + k - 1) \quad (29)$$

for each $\mathcal{E} = S_{\lambda} \mathcal{S}^* \subset \mathcal{G}$. By the condition $\kappa_{\mathcal{E}} \leq \kappa_{\mathcal{G}}$ with (24) and the assumption $l \geq k + 1$ for $A + B > 0$, we obtain a necessary condition

$$\varphi_{\mathcal{E}}(k) \leq d \quad (30)$$

for any irreducible component \mathcal{E} in \mathcal{G} for $A + B = 0$ or 1.

Lemma 2.10. *For a sufficiently large k , there is no \mathcal{F} with $A + B = 1$.*

Proof. Note that we may assume $\mathcal{G} \neq 0$. In fact, if $\mathcal{G} = 0$, (21) gives the condition $l(k - 1) = d$ and hence $k \leq \sqrt{d + 1}$.

For $(A, B) = (0, 1)$, we have $0 < \kappa_{\mathcal{E}} \leq -k + d + 1$ for all irreducible $\mathcal{E} \subset \mathcal{G}$ from (30). Hence $k \leq d$ holds.

For $(A, B) = (1, 0)$, we also have

$$\kappa_{\mathcal{E}} \leq k + d + 1 \quad (31)$$

from (30).

Let us assume $\mathcal{F}^{\mathcal{S}} \neq 0$, first. For $\mathcal{E} \subset \mathcal{F}^{\mathcal{S}}$,

$$|\lambda| \leq 1 + \frac{k + d + 1}{\text{rank } \mathcal{E}} \quad (32)$$

holds from (23). If $k > d + 1$, this implies $|\lambda| \leq 2$ since $\text{rank } \mathcal{E} \geq k$. Since $\mathcal{E} \neq \mathcal{S}^*$, we have $|\lambda| = 2$ and hence $\mathcal{E} = \wedge^2 \mathcal{S}^*$ or $\text{Sym}^2 \mathcal{S}^*$. From (31) and (23), we have

$$2k > k + d + 1 \geq \kappa_{\mathcal{E}} \geq \text{rank } \mathcal{E} \geq \binom{k}{2} \quad (33)$$

if $k > d + 1$. By solving it, we obtain $k \leq 4$. Thus we have $k \leq \max\{d + 1, 4\}$ if $\mathcal{F}^{\mathcal{S}} \neq 0$.

In the remaining case $\mathcal{G} = \mathcal{F}^{\text{line}} \neq 0$, we can also see that $k \leq \frac{1}{2}(1 + \sqrt{4d + 9})$ by $\text{rank } \mathcal{F}^{\text{line}} \leq c_1(\mathcal{F}^{\text{line}})$ with (21), (22) and $l \geq k + 1$. □

Finally we consider the most intricate case with $(A, B) = (0, 0)$, i.e. $\mathcal{F}^{\mathcal{Q}} = 0$. Contrary to Lemmas 2.6, 2.8 and 2.10, we should allow $n = 2k$ in this case (recall that (A5) is the condition that $\mathcal{F}^{\mathcal{Q}} = 0$ if $n = 2k$).

Lemma 2.11. *Assume $\mathcal{F}^{\mathcal{Q}} = 0$. For a sufficiently large k , any irreducible component $\mathcal{E} \subset \mathcal{F}^{\mathcal{S}}$ is $\wedge^2 \mathcal{S}^*$, $\text{Sym}^2 \mathcal{S}^*$, $\wedge^{k-1} \mathcal{S}^*$, or $\mathcal{S}^*(1)$.*

Proof. We can show this lemma by the case analysis similar to (Qi)'s in the proof of Lemma 2.6. Let $\mathcal{E} = S_{\lambda} \mathcal{S}^* \subset \mathcal{F}^{\mathcal{S}}$ be an irreducible component which is not $\wedge^2 \mathcal{S}^*$, $\text{Sym}^2 \mathcal{S}^*$, $\wedge^{k-1} \mathcal{S}^*$, nor $\mathcal{S}^*(1)$, i.e. $\lambda \neq (1, 1, 0, \dots, 0), (2, 0, \dots, 0), (1, \dots, 1, 0), (2, 1, \dots, 1)$. In the following table, we give the lower bounds of $\varphi_{\mathcal{E}}(k)$ for each cases (S1), \dots , (S7), for which we use (23), (29) and the similar evaluation of $\text{rank } \mathcal{E}$ as the proof of Lemma 2.6. This implies the boundedness of k by the condition (30). Note that here we define the i -th case (Si) not including any λ in (S1), \dots , (S($i-1$)) for all i , similarly to (Qi)'s.

conditions for $\mathcal{E} = S_{\lambda} \mathcal{S}^*$	lower bounds of $\varphi_{\mathcal{E}}(k)$
(S1) $\lambda_i > \lambda_{i+1}$ and $\lambda_j > \lambda_{j+1}$ for some $i \neq j$, and $k \geq 3$	$k^2 - 2$
(S2) $\lambda_i - \lambda_{i+1} \geq 2$ for some $2 \leq i \leq k-2$, and $k \geq 4$	$\frac{1}{4}k^2(k^2 - 5)$
(S3) $\lambda_{k-1} - \lambda_k \geq 2$, and $k \geq 3$	$\frac{1}{2}k(2k^2 - 3k - 3)$
(S4) $\lambda_1 - \lambda_2 \geq 3$	$\frac{1}{3}k(k^2 + 2)$
(S5) $\lambda_i - \lambda_{i+1} = 1$ for some $3 \leq i \leq k-3$, and $k \geq 6$	$\frac{1}{3}k(k^2 - 6k + 2)$
(S6) $\lambda_{k-2} - \lambda_{k-1} = 1$ and $k \geq 5$	$\frac{1}{2}k(k^2 - 6k + 3)$
(S7) $\lambda_k \geq 1$ with $ \lambda \neq k+1$	$\frac{1}{2}k(k^2 - 2k - 1)$

□

Proof of Proposition 2.2. By Lemma 2.4, it suffices to show the finiteness of \mathcal{F} which satisfies (2) and (A1)-(A5). By Lemmas 2.6 and 2.8 to 2.10, we see the finiteness of such \mathcal{F} with $\mathcal{F}^{\mathcal{Q}} \neq 0$.

By Lemmas 2.9 and 2.11, it is enough to show that for sufficiently large k , there is no \mathcal{F} such that $\mathcal{F}^{\mathcal{Q}} = 0$ and $\mathcal{F}^{\mathcal{S}}$ has the form

$$\mathcal{F}^{\mathcal{S}} = (\wedge^2 \mathcal{S}^*)^{\oplus \alpha} \oplus (\text{Sym}^2 \mathcal{S}^*)^{\oplus \beta} \oplus (\wedge^{k-1} \mathcal{S}^*)^{\oplus \gamma} \oplus (\mathcal{S}^*(1))^{\oplus \delta}, \quad (34)$$

where α, β, γ and δ are non-negative integers. In fact, we show that there is no such $\mathcal{F}^{\mathcal{S}}$ if $k > \max\{3 + \sqrt{9 + 2d}, d + 2\}$ as follows.

For such $\mathcal{F}^{\mathcal{S}}$, we have

$$\text{rank } \mathcal{F}^{\mathcal{S}} = \alpha \binom{k}{2} + \beta \binom{k+1}{2} + \gamma k + \delta k, \quad (35)$$

$$c_1(\mathcal{F}^{\mathcal{S}}) = \alpha(k-1) + \beta(k+1) + \gamma(k-1) + \delta(k+1), \quad (36)$$

$$\kappa_{\mathcal{F}^{\mathcal{S}}} = \alpha \binom{k}{2} + \beta \binom{k+1}{2} + \gamma k(k-2) + \delta k^2. \quad (37)$$

By using the condition $\kappa_{\mathcal{F}^{\mathcal{S}}} \leq \kappa_{\mathcal{F}} = k^2 + d$, we obtain

$$(\alpha + \beta + 2\gamma + 2\delta - 2)k^2 + (-\alpha + \beta - 4\gamma)k \quad (38)$$

$$= (\alpha + \beta + 2\gamma + 2\delta - 2)k(k-2) + (\alpha + 3\beta + 4\delta - 4)k \leq 2d. \quad (39)$$

If $\alpha + \beta + 2\gamma + 2\delta > 2$, we have $k(k-2) - 4k \leq 2d$, i.e. $k \leq 3 + \sqrt{9+2d}$. Thus it holds that $\alpha + \beta + 2\gamma + 2\delta \leq 2$ if $k > 3 + \sqrt{9+2d}$, and hence $(\alpha, \beta, \gamma, \delta)$ is one of the following; $(2, 0, 0, 0), (1, 1, 0, 0), (0, 2, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, 0, 0), (0, 1, 0, 0)$ and $(0, 0, 0, 0)$.

Since $\mathcal{F} = \mathcal{F}^{\mathcal{S}} \oplus \mathcal{F}^{\text{line}}$, it holds that

$$\text{rank } \mathcal{F}^{\text{line}} = kl - \text{rank } \mathcal{F}^{\mathcal{S}} - d, \quad (40)$$

$$c_1(\mathcal{F}^{\text{line}}) = n - c_1(\mathcal{F}^{\mathcal{S}}). \quad (41)$$

Now we have a condition $\text{rank } \mathcal{F}^{\text{line}} \leq c_1(\mathcal{F}^{\text{line}})$. Hence

$$k^2 - 2k - \text{rank } \mathcal{F}^{\mathcal{S}} + c_1(\mathcal{F}^{\mathcal{S}}) \leq d \quad (42)$$

holds since $k^2 - 2k \leq kl - n$ by (A2). Thus we have

$$(2 - \alpha - \beta)k^2 + (-4 + 3\alpha + \beta)k + 2(-\alpha + \beta - \gamma + \delta) \leq 2d. \quad (43)$$

If $k > d + 2$, we see that (43) is satisfied only for $(\alpha, \beta, \gamma, \delta) = (1, 1, 0, 0)$ or $(0, 2, 0, 0)$.

Finally, we use the invariant κ . It holds that

$$\kappa_{\mathcal{F}^{\text{line}}} = \kappa_{\mathcal{F}} - \kappa_{\mathcal{F}^{\mathcal{S}}} = k^2 + d - \kappa_{\mathcal{F}^{\mathcal{S}}} = \begin{cases} d & \text{if } (\alpha, \beta, \gamma, \delta) = (1, 1, 0, 0) \\ d - k & \text{if } (\alpha, \beta, \gamma, \delta) = (0, 2, 0, 0). \end{cases} \quad (44)$$

If $\kappa_{\mathcal{F}^{\text{line}}} = 0$ and $k > d$, (44) leads to a contradiction. If $\kappa_{\mathcal{F}^{\text{line}}} \neq 0$, we have $\kappa_{\mathcal{F}^{\text{line}}} \geq k - 1$ since $\kappa_{\mathcal{O}(p)} = pk - 1 \geq k - 1$ for $p > 0$. Hence (44) leads to a contradiction for $k > d + 1$.

Therefore there exists no choice of \mathcal{F} for a sufficiently large k . This completes the proof. \square

3. COMPLETE INTERSECTION CALABI–YAU 3-FOLDS

Let us apply our method to the classification of complete intersection Calabi–Yau 3-folds and show Theorem 1.1.

Remark 3.1. In order to create Table 1, we omit some duplication originated from several kinds of natural identifications.

There are the natural isomorphisms between $G(k, n)$ and $G(n - k, n)$, $Z_{\mathcal{S}^*} \subset G(k, n)$ and $G(k, n - 1)$, $Z_{\mathcal{Q}} \subset G(k, n)$ and $G(k - 1, n - 1)$, and $Z_{\wedge^2 \mathcal{Q}} \subset G(k, 2k)$ and $Z_{\wedge^2 \mathcal{Q}} \subset G(k, 2k + 1)$ discussed in Section 2.

We have one more identification, which was not considered in Section 2. On $LG(k, 2k) \simeq Z_{\wedge^2 \mathcal{S}^*} \subset G(k, 2k)$, $\mathcal{S}^*|_{Z_{\wedge^2 \mathcal{S}^*}}$ and $\mathcal{Q}|_{Z_{\wedge^2 \mathcal{S}^*}}$ are isomorphic. Hence we identify $Z_{\wedge^2 \mathcal{S}^* \oplus S_{\lambda} \mathcal{S}^* \oplus \mathcal{F}'}$ and $Z_{\wedge^2 \mathcal{S}^* \oplus S_{\lambda} \mathcal{Q} \oplus \mathcal{F}'}$ $G(k, 2k)$.

Let $\mathcal{F} = \mathcal{F}^{\mathcal{S}} \oplus \mathcal{F}^{\mathcal{Q}} \oplus \mathcal{F}^{\text{line}}$ be a homogeneous vector bundle on a Grassmannian $G(k, n)$ which satisfies

$$\text{rank } \mathcal{F} = k(n - k) - 3 \quad \text{and} \quad c_1(\mathcal{F}) = n \quad (45)$$

and the assumptions (A1)-(A5).

First we consider the case $\mathcal{F}^{\mathcal{Q}} \neq 0$. If $\mathcal{F}^{\mathcal{Q}} \neq (\wedge^{l-1} \mathcal{Q})^{\oplus A} \oplus (\mathcal{Q}(1))^{\oplus B}$, we obtain two Calabi–Yau 3-folds, $(\alpha 2)$ and $(\beta 2)$ from Lemma 2.6, i.e. No. 29 and 5 in Table 1, respectively.

Next, we consider the case with $\mathcal{F}^{\mathcal{Q}} = (\wedge^{l-1} \mathcal{Q})^{\oplus A} \oplus (\mathcal{Q}(1))^{\oplus B}$. From Lemma 2.7, the possible values are $(A, B) = (2, 0), (1, 0), (0, 1)$ or $(0, 0)$. For $(A, B) = (2, 0)$, Lemma 2.8 gives a Calabi–Yau 3-fold $(\gamma 3)$ i.e. No. 24. For $A + B = 1$, there is a bound $k \leq 4$ as in the proof of Lemma 2.10. For each $k \leq 4$, we use the other condition (25) and obtain seven Calabi–Yau 3-folds, No. 10, 11, 15 to 18 and 25.

Let $\mathcal{F}^\mathcal{Q} = 0$. Suppose that $k \leq 5$, first. From

$$(|\lambda| - 1)k \leq (|\lambda| - 1) \operatorname{rank} \mathcal{E} = \kappa_{\mathcal{E}} \leq \kappa_{\mathcal{F}} = k^2 + 3 \quad (46)$$

for an irreducible bundle $\mathcal{E} = S_\lambda \mathcal{S}^* \subset \mathcal{F}^\mathcal{S}$, we only need to check the Young diagrams with $|\lambda| \leq 4$ for $k = 2$, $|\lambda| \leq 5$ for $k = 3, 4$ and $|\lambda| \leq 6$ for $k = 5$. We restrict the possible bundles further by the actual evaluation of $\kappa_{\mathcal{E}} \leq k^2 + 3$. For instance, if $k = 2$, the possible form becomes

$$\mathcal{F} = \mathcal{S}^*(1)^{\oplus x_1} \oplus (\operatorname{Sym}^2 \mathcal{S}^*)^{\oplus x_2} \oplus \mathcal{O}(1)^{\oplus y_1} \oplus \mathcal{O}(2)^{\oplus y_2} \oplus \mathcal{O}(3)^{\oplus y_3} \oplus \mathcal{O}(4)^{\oplus y_4} \quad (47)$$

The conditions (45) turn into

$$7 = \kappa_{\mathcal{F}} = 4x_1 + 3x_2 + y_1 + 3y_2 + 5y_3 + 7y_4, \quad (48)$$

$$4 \leq n = 3x_1 + 3x_2 + y_1 + 2y_2 + 3y_3 + 4y_4. \quad (49)$$

By solving them, we get the remaining Calabi–Yau 3-folds among No. 1–No. 18. The same argument works for each $k \leq 5$ and we obtain the remaining cases among No. 19–No. 33.

Assume $\mathcal{F}^\mathcal{Q} = 0$ and $k \geq 6$. In this case, $\mathcal{F}^\mathcal{S} = (\wedge^2 \mathcal{S}^*)^{\oplus \alpha} \oplus (\operatorname{Sym}^2 \mathcal{S}^*)^{\oplus \beta} \oplus (\wedge^{k-1} \mathcal{S}^*)^{\oplus \gamma} \oplus (\mathcal{S}^*(1))^{\oplus \delta}$ for some α, β, γ and δ by Lemma 2.11. By the proof of Proposition 2.2, we have

$$k \leq \max\{3 + \sqrt{9 + 2d}, d + 2\} = \max\{3 + \sqrt{15}, 5\} < 7, \quad (50)$$

and hence $k = 6$. We note that (39), (43) and (44) hold even if $k \leq \max\{3 + \sqrt{9 + 2d}, d + 2\}$. By equalities (39), (43) for $d = 3, k = 6$, we see that $(\alpha, \beta, \gamma, \delta)$ must be $(1, 1, 0, 0)$. Hence $\kappa_{\mathcal{F}^\text{line}} = d = 3$ by (44). However there is no such \mathcal{F}^line since $\kappa_{\mathcal{O}(p)} = pk - 1 \geq 5 > 3$ for $p > 0$. Thus there is no solution for $k \geq 6$.

Finally, we consider \mathcal{F} which does not satisfy (A5), i.e. $\mathcal{F}^\mathcal{Q} \neq 0$ with $n = 2k$. Let \mathcal{F} be a classified homogeneous vector bundle over $G(k, 2k)$ which satisfies (45) and $\mathcal{F}^\mathcal{Q} = 0$. As in Example 2.5, we can obtain another \mathcal{F}' on $G(k, 2k)$ by replacing one of $S_\lambda \mathcal{S}^* \subset \mathcal{F}$ with $S_\lambda \mathcal{Q}$. If $\mathcal{F}^\mathcal{S}$ is irreducible, such \mathcal{F}' does not give a new family, since $Z_{S_\lambda \mathcal{S}^* \oplus \mathcal{F}^\text{line}}$ and $Z_{S_\lambda \mathcal{Q} \oplus \mathcal{F}^\text{line}}$ in $G(k, 2k)$ are identified. Otherwise, i.e. for No. 21, 31 and 33, $\mathcal{F}^\mathcal{S}$ contains $\wedge^2 \mathcal{S}^*$ as an irreducible component. Hence we do not obtain a new family in these cases as well because of the last identification in Remark 3.1. Thus we do not have \mathcal{F} with $\mathcal{F}^\mathcal{Q} \neq 0$ and $n = 2k$ in Table 1.

The Hodge numbers for Calabi–Yau 3-folds can be calculated by the similar way in [Küc95, Section 4.4]. All examples except for No. 30 have $h^{0,0}(Z_{\mathcal{F}}) = 1$, i.e. they are irreducible 3-folds. For No. 30, we have $h^{0,0}(Z_{\mathcal{F}}) = 2$, which is consistent with the description in Section 7. We also have $h^1(\mathcal{O}_{Z_{\mathcal{F}}}) = 0$ for all examples except for No. 26. This means that they are Calabi–Yau 3-folds in the strict sense, while No. 26 is an abelian 3-fold as we see in Section 7.

4. ALTERNATIVE DESCRIPTION: NO. 4, 5, 7 AND 10

In the rest of this paper, we study alternative descriptions of some of Calabi–Yau 3-folds in Table 1. In this section, we treat No. 4, 5, 7 and 10.

Let W be a linear space of dimension n , and let \mathcal{E} be a globally generated locally free sheaf on $G(k, W)$. We consider a description of $Z_{\mathcal{E}(1)} \subset G(k, W)$.

We denote by \mathcal{K} the kernel of the natural surjection $H^0(\mathcal{E}) \otimes \mathcal{O} \rightarrow \mathcal{E}$. We consider the following diagram:

$$\begin{array}{ccc} & \mathbb{P} := \mathbb{P}_{G(k,W)}(\mathcal{K} \oplus \mathcal{O}(-1)) & \\ \mu \swarrow & & \searrow \pi \\ \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W) & & G(k, W), \end{array} \quad (51)$$

where π is the projective bundle induced by $\mathcal{K} \oplus \mathcal{O}(-1)$ and μ is the morphism induced by $\mathcal{K} \oplus \mathcal{O}(-1) \subset (H^0(\mathcal{E}) \oplus \wedge^k W) \otimes \mathcal{O}_{G(k,W)}$. Set

$$\Sigma = \mu(\mathbb{P}) \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W).$$

Since $\mathcal{O}(1)$ is very ample on $G(k, W)$,

$$\mu : \mathbb{P}_{G(k,W)}(\mathcal{K} \oplus \mathcal{O}(-1)) \setminus \mathbb{P}_{G(k,W)}(\mathcal{K} \oplus \{0\}) \rightarrow \Sigma \setminus (\Sigma \cap \mathbb{P}(H^0(\mathcal{E})))$$

is an isomorphism for $\mathbb{P}(H^0(\mathcal{E})) = \mathbb{P}(H^0(\mathcal{E}) \oplus \{0\}) \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)$.

Let \bar{s} be an element in $H^0(\mathcal{E}) \otimes (\wedge^k W)^*$. In other words, \bar{s} is a global section of $H^0(\mathcal{E}) \otimes \mathcal{O}(1)$ on $G(k, W)$. Let $s \in H^0(\mathcal{E}(1))$ be the image of \bar{s} by the natural map $H^0(\mathcal{E}) \otimes H^0(\mathcal{O}(1)) \rightarrow H^0(\mathcal{E}(1))$.

Since $\bar{s} \in H^0(\mathcal{E}) \otimes (\wedge^k W)^*$, we have a linear map $\wedge^k W \rightarrow H^0(\mathcal{E})$, which we also denote by the same letter \bar{s} . Hence we have a linear embedding

$$\mathbb{P}(\wedge^k W) \hookrightarrow \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W) \quad : \quad [p] \mapsto [(\bar{s}(p), p)]. \quad (52)$$

Let $P_{\bar{s}} \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)$ be the image of the embedding. In other words, $P_{\bar{s}}$ is the linear subvariety of codimension $h^0(\mathcal{E})$ cut out by the image of

$$(\text{id}_{H^0(\mathcal{E})^*}, \bar{s}^*) : H^0(\mathcal{E})^* \hookrightarrow H^0(\mathcal{E})^* \oplus (\wedge^k W)^* = H^0(\mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W), \mathcal{O}(1)). \quad (53)$$

Conversely, if a linear subvariety $P \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)$ of codimension $h^0(\mathcal{E})$ satisfies $P \cap \mathbb{P}(H^0(\mathcal{E})) = \emptyset$, there exists a section $\bar{s} \in H^0(\mathcal{E}) \otimes \mathcal{O}(1)$ such that $P = P_{\bar{s}}$. Hence $P_{\bar{s}} \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)$ is general if so is \bar{s} .

Proposition 4.1. *Let \bar{s} be a general element in $H^0(\mathcal{E}) \otimes (\wedge^k W)^*$, and let $s, P_{\bar{s}}$ be as above. Let $Z \subset G(k, W)$ be the zero locus of the section $s \in H^0(\mathcal{E}(1))$ and let $Z' \subset \Sigma$ be the linear section of Σ by $P_{\bar{s}}$. Then $Z \subset \mathbb{P}(\wedge^k W)$ and $Z' \subset P_{\bar{s}}$ are projectively equivalent.*

To prove Proposition 4.1, we use the following lemma. In the lemma, we consider Grassmannians of *quotient spaces*: For a vector space E , we denote by $G(E, r)$ the Grassmannian of r -dimensional quotient spaces of E . More generally, for a coherent sheaf \mathcal{E} on a noetherian scheme S , we set a scheme $G_S(\mathcal{E}, r)$ over S by

$$G_S(\mathcal{E}, r) := \text{Quot}_{\mathcal{E}/S/S}^{r, \mathcal{O}_S},$$

which parametrizes locally free quotient sheaves of $\varphi^* \mathcal{E}$ of rank r for each $\varphi : T \rightarrow S$ (see [Gro95], [Nit05, 5.1.5]). In particular, the fiber of $G_S(\mathcal{E}, r) \rightarrow S$ over $s \in S$ is the Grassmannian $G(\mathcal{E} \otimes k(s), r)$. If \mathcal{E} is locally free of rank n , we call $G_S(\mathcal{E}, r) \rightarrow S$ a $G(n, r)$ -bundle over S .

Lemma 4.2. *Let $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ be a homomorphism between locally free sheaves on a variety X . Let $0 \leq r \leq f - 1$ be a non-negative integer for $f = \text{rank } \mathcal{F}$ and let $\pi : G_X(\mathcal{F}, f - r) \rightarrow X$ be the Grassmannian over X . Let Y be the zero locus of*

$$\pi^* \mathcal{E} \xrightarrow{\pi^* \varphi} \pi^* \mathcal{F} \rightarrow \mathcal{Q}_\pi,$$

where \mathcal{Q}_π is the tautological quotient bundle of rank $f - r$ on $G_X(\mathcal{F}, f - r)$. Let $Z_\varphi(i) \subset X$ be the i -th degeneracy locus of φ , i.e. the locus where the rank of φ is at most i . Then

- (i) $\pi(Y)$ is contained in $Z_\varphi(r)$,
- (ii) for each $0 \leq i \leq r$, $\pi : Y \rightarrow Z_\varphi(r)$ is a $G(f-i, f-r)$ -bundle over $Z_\varphi(i) \setminus Z_\varphi(i-1)$, where we set $Z_\varphi(-1) = \emptyset$.

Proof. Let $\text{coker } \varphi$ be the cokernel of φ . By the canonical quotient homomorphism $\mathcal{F} \rightarrow \text{coker } \varphi$, $G_X(\text{coker } \varphi, f-r)$ is embedded into $G_X(\mathcal{F}, f-r)$. Then $G_X(\text{coker } \varphi, f-r)$ is the locus where $\pi^*\mathcal{F} \rightarrow \mathcal{Q}_\pi$ factors through $\pi^*\text{coker } \varphi$, and the locus is nothing but the zero locus Y of $\pi^*\mathcal{E} \rightarrow \pi^*\mathcal{F} \rightarrow \mathcal{Q}_\pi$. Hence $G_X(\text{coker } \varphi, f-r) \subset G_X(\mathcal{F}, f-r)$ coincides with Y . Thus the fiber of $Y = G_X(\text{coker } \varphi, f-r) \rightarrow X$ over $x \in X$ is the Grassmannian $G(\text{coker } \varphi_x, f-r)$. Hence $(\pi|_Y)^{-1}(x)$ is empty if $\dim \text{coker } \varphi_x < f-r$, which is equivalent to $x \notin Z_\varphi(r)$. Thus (i) holds.

By the definition of degeneracy loci, $\text{coker } \varphi$ is locally free of rank $f-i$ on $Z_\varphi(i) \setminus Z_\varphi(i-1)$. Hence (ii) follows. \square

Proof of Proposition 4.1. We have the following commutative diagram on $G(k, W)$

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{K} \oplus \mathcal{O}(-1) & \xrightarrow{p_2} & \mathcal{O}(-1) \longrightarrow 0 \\
& & \parallel & & \downarrow (\iota, \bar{s}) & & \downarrow s \\
0 & \longrightarrow & \mathcal{K} & \xrightarrow{\iota} & H^0(\mathcal{E}) \otimes \mathcal{O} & \longrightarrow & \mathcal{E} \longrightarrow 0,
\end{array} \tag{54}$$

where ι is the natural injection and p_2 is the projection to the second factor.

We apply Lemma 4.2 to $(\iota, \bar{s})^* : H^0(\mathcal{E})^* \otimes \mathcal{O} \rightarrow \mathcal{K}^* \oplus \mathcal{O}(1)$ and $r = \text{rank } \mathcal{K}$. In this case,

$$G_X(\mathcal{F}, f-r) = G_{G(k, W)}(\mathcal{K}^* \oplus \mathcal{O}(1), 1) = \mathbb{P}_{G(k, W)}(\mathcal{K} \oplus \mathcal{O}(-1))$$

and Y is the zero locus of

$$H^0(\mathcal{E})^* \otimes \mathcal{O}_\mathbb{P} \rightarrow \pi^*(\mathcal{K}^* \oplus \mathcal{O}_{G(k, W)}(1)) \rightarrow \mathcal{O}_\pi(1), \tag{55}$$

where $\mathcal{O}_\pi(-1)$ is the tautological invertible sheaf of the projective bundle π . By the diagram (54), it holds that $\text{coker}(i, \bar{s})^* \simeq \text{coker } s^*$ for $s^* : \mathcal{E}^* \rightarrow \mathcal{O}(1)$. Hence we have $Z_{(i, \bar{s})^*}(\text{rank } \mathcal{K}) = Z_{s^*}(0) = Z$ and $Z_{(i, \bar{s})^*}(\text{rank } \mathcal{K} - 1) = Z_{s^*}(-1) = \emptyset$. Thus $\pi : Y \rightarrow Z$ is an isomorphism by Lemma 4.2.

By construction, (55) is the pullback of

$$H^0(\mathcal{E})^* \otimes \mathcal{O}_{\mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)} \rightarrow \mathcal{O}_{\mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)}(1)$$

induced from (53) by μ . Hence $Y = \mu^{-1}(P_{\bar{s}})$ holds. Since $\mu : \mathbb{P} \rightarrow \Sigma \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)$ is an isomorphism over $\Sigma \setminus (\Sigma \cap \mathbb{P}(H^0(\mathcal{E})))$ and $P_{\bar{s}}$ does not intersect $\mathbb{P}(H^0(\mathcal{E})) \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)$, we have an isomorphism $\mu|_Y : Y \rightarrow \Sigma \cap P_{\bar{s}} = Z'$.

Hence both Z and Z' are isomorphic to Y . Since $Z \xrightarrow{\sim} Y \xrightarrow{\sim} Z' \subset \Sigma$ is nothing but the restriction of $\mathbb{P}(\wedge^k W) \xrightarrow{\sim} P_{\bar{s}} \subset \mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^k W)$ in (52), Z and Z' are projectively equivalent. \square

4.1. No. 4 and 7. In this subsection, we consider the case $\mathcal{E} = \mathcal{S}^*$ on $G(2, W)$ for $5 \leq \dim W \leq 8$. In this case, $H^0(\mathcal{E}) = W^*$, $\mathcal{K} = \mathcal{Q}^*$, and Σ is contained in $\mathbb{P}(W^* \oplus \wedge^2 W)$.

Lemma 4.3. *Suppose that $\dim W = 5, 6, 7$ or 8 . The variety $\Sigma \subset \mathbb{P}(W^* \oplus \wedge^2 W)$ is projectively equivalent to a Schubert variety of a generalized Grassmannian $G/P = OG(5, 10) = D_5/P_5$, $\mathbb{O}\mathbb{P}^2 = E_6/P_6$, E_7/P_7 or $E_8/P_8 \subset \mathbb{P}(H^0(\mathcal{O}_{G/P}(1)))$, respectively.*

Proof. We recall the Tits transform. Let G be a simple Lie group and fix a Borel subgroup $B \subset G$. Consider the diagram

$$\begin{array}{ccc} & G/(P \cap Q) & \\ \tilde{\mu} \swarrow & & \searrow \tilde{\pi} \\ G/Q & & G/P \end{array} \quad (56)$$

where P and Q are distinct parabolic subgroups of G containing B . For a subset $X \subset G/P$, the *Tits transform* of X is defined by $\mathcal{T}(X) := \tilde{\mu}(\tilde{\pi}^{-1}(X)) \subset G/Q$, (see [LM04] for detail).

Let Δ be the set of simple roots of G and $\Delta' \subset \Delta$ a subset of simple roots which defines a semi-simple Lie subgroup $G' \subset G$. From the discussion of [LM04, Section 2.7.1], a homogeneously embedded homogeneous submanifold $G'/(Q \cap G') \subset G/Q$ is a smooth Schubert variety of G/Q which coincides with the Tits transform of the Borel fixed point $o := P/P$ in G/P , where P is the parabolic subgroup whose Lie algebra $\mathfrak{p} = \text{Lie } P$ satisfies $\Delta' = \{\alpha \in \Delta \mid \mathfrak{g}_{-\alpha} \subset \mathfrak{p}\}$. In particular, we get the following diagram for $G = D_5, E_6, E_7$ and E_8 and $G' = A_{n-1}$, where $n = \text{rank } G$ and the crossed Dynkin diagrams and the encircled crossed Dynkin diagram represent flag manifolds and the corresponding homogeneously embedded homogeneous submanifold, respectively.

$$\begin{array}{ccccc} & \text{Dynkin diagram 1} & & \text{Dynkin diagram 2} & \\ & \swarrow \tilde{\mu} & & \searrow \tilde{\pi} & \\ \text{Dynkin diagram 3} & & \mathbb{P}_{G(2,n)}(\mathcal{G}|_{G(2,n)}) & & G(2,n) \\ & \swarrow \tilde{\mu} & & \searrow \tilde{\pi} & \\ \mathcal{T}(G(2,n)) & & \text{Dynkin diagram 4} & & o \end{array} \quad (57)$$

Let us use the numeration $1, 2, 3, 5, \dots, n$ for the nodes in the upper row and 4 for the unique lower node in the Dynkin diagram and denote by ϖ_i and P_i the fundamental weight and the maximal parabolic subgroup which correspond to the node i , respectively. Let \mathcal{G} be the homogeneous vector bundle corresponding to the fundamental representation V_{ϖ_n} , i.e. $\tilde{\pi} : G/(P_2 \cap P_n) \rightarrow G/P_2$ is nothing but the \mathbb{P}^{n-2} -bundle $\mathbb{P}_{G/P_2}(\mathcal{G}) \rightarrow G/P_2$, and $\tilde{\mu}$ is the morphism induced by the tautological line bundle of $\mathbb{P}_{G/P_2}(\mathcal{G})$ in the diagram (57). Here the restriction $\mathcal{G}|_{G(2,n)}$ is isomorphic to $\mathcal{Q}^* \oplus \mathcal{O}(-1)$ on $G(2,n) \subset G/P_2$ from the following branching rules of the representations of parabolic subgroups $P_2 \cap A_{n-1} \subset P_2$,

$$\begin{array}{c} 0 \ 0 \ 0 \ 0 \ 0 \ 1 \\ \times \\ 0 \end{array} = \begin{array}{c} 0 \ 0 \ 0 \ 0 \ 0 \ 1 \\ \times \\ 0 \end{array} \oplus \begin{array}{c} 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\ \times \\ 0 \end{array} \quad (58)$$

where the crossed Dynkin diagram and the encircled crossed Dynkin diagrams represent the parabolic subgroup P_2 and $P_2 \cap A_{n-1}$, respectively, and the integers on each nodes are coefficients of fundamental weights of the highest weight of the irreducible representations. Hence the restriction of the left side of the diagram (57) coincides with (51) for $k = 2, \dim W = n, \mathcal{E} = \mathcal{S}^*, \mathcal{K} = \mathcal{Q}^*$, and then Σ coincides with the Tits transform $\mathcal{T}(G(2,n))$, which is a Schubert variety of G/P_n by [CR13, Lemma 2.4]. \square

Remark 4.4. By [CR13, Lemma 2.4], we can compute the Tits transform $\mathcal{T}(G(2, n)) \subset G/P_n$, explicitly. For each $5 \leq n \leq 8$, the list of reduced expressions of the corresponding elements in the Weyl groups are given in the following table, where $i_1 \cdots i_m$ represents an element $w = s_{i_1} \cdots s_{i_m}$ of the set of minimal length representatives $W_{\min}^{P_j} \subset W_G$ of the cosets W_G/W_{P_j} for $j = 2$ or n with simple reflections s_1, \dots, s_n in the Weyl group W_G .

	$w \in W_{\min}^{P_n}$ with $\Sigma = \overline{Bw^{-1}P_n/P_n}$	$w \in W_{\min}^{P_2}$ with $G(2, n) = \overline{Bw^{-1}P_2/P_2}$
$G = D_5$	532143253	213253
E_6	653214325365	21325365
E_7	765321432536576	2132536576
E_8	876532143253657687	213253657687

From Proposition 4.1 and Lemma 4.3 with the above computation, we get the description of No. 4 and 7 in Table 1.

For $n = 5$, Σ is the Schubert divisor of $OG(5, 10)$, as shown by [CCGK16, Section 16] in a different way. Especially, a Calabi–Yau 3-fold in No. 4 is isomorphic to the complete intersection of $OG(5, 10)$ by six hyperplanes and a quadric.

In the case of $n = 6$, Σ is the 12-dimensional Schubert variety of degree 33 in the Cayley plane $\mathbb{O}\mathbb{P}^2$, which is first pointed out by [Gal14, GKM]. Therefore a Calabi–Yau 3-fold in No. 7 is isomorphic to the complete intersection of the 12-dimensional Schubert variety of degree 33 in $\mathbb{O}\mathbb{P}^2$ by nine hyperplanes.

4.2. No. 5. We apply Proposition 4.1 to $\mathcal{E} = \wedge^2 \mathcal{Q}$ on $G(2, W)$ for $\dim W = 5$. Then we have a subvariety Σ in $\mathbb{P}(H^0(\mathcal{E}) \oplus \wedge^2 W) = \mathbb{P}(\wedge^2 W \oplus \wedge^2 W) = \mathbb{P}^{19}$.

Lemma 4.5. *It holds that*

$$\Sigma = \{[q, p] \mid p \wedge p = q \wedge p = 0 \in \wedge^4 W\}_{red} \subset \mathbb{P}(\wedge^2 W \oplus \wedge^2 W),$$

where X_{red} is the scheme with the reduced structure for a scheme X .

Proof. In this case, \mathcal{K} is the kernel of $\wedge^2 W \otimes \mathcal{O} \rightarrow \wedge^2 \mathcal{Q}$. Take a point $x \in G(2, W) \subset \mathbb{P}(\wedge^2 W)$ and let $p \in \wedge^2 W$ be a corresponding element. Then the fiber $\pi^{-1}(x)$ is $\mathbb{P}(\mathcal{K}_x \oplus \mathbb{C}p)$. For $q \in \wedge^2 W$, q is contained in $\mathcal{K}_x \subset \wedge^2 W$ if and only if $q \wedge p = 0$. Hence it holds that

$$\Sigma = \{[q, tp] \in \mathbb{P}(\wedge^2 W \oplus \wedge^2 W) \mid [p] \in G(2, W), t \in \mathbb{C}, q \wedge p = 0\}.$$

In particular, Σ is defined by $p \wedge p = q \wedge p = 0$ scheme-theoretically outside $\mathbb{P}(\wedge^2 W \oplus \{0\})$.

On the other hand, $[q, 0] \in \mathbb{P}(\wedge^2 W \oplus \{0\})$ is contained in Σ if and only if there exists $[p] \in G(2, W)$ such that $q \wedge p = 0$. This condition is equivalent to $G(2, W) \cap \mathbb{P}(\ker(q \wedge)) \neq \emptyset$, where $\ker(q \wedge)$ is the kernel of the linear map $q \wedge : \wedge^2 W \rightarrow \wedge^4 W$. Since $\dim G(2, W) = 6$ and

$$\dim \ker(q \wedge) \geq \dim \wedge^2 W - \dim \wedge^4 W = 5,$$

$G(2, W) \cap \mathbb{P}(\ker(q \wedge)) \subset \mathbb{P}(\wedge^2 W) = \mathbb{P}^9$ is non-empty. Hence $\mathbb{P}(\wedge^2 W \oplus \{0\})$ is contained in Σ and we have

$$\Sigma = \{[q, p] \in \mathbb{P}(\wedge^2 W \oplus \wedge^2 W) \mid p \wedge p = q \wedge p = 0\}$$

as subsets in $\mathbb{P}(\wedge^2 W \oplus \wedge^2 W)$. □

Consider a family $\{\Sigma_t\}_{t \in \mathbb{C}}$ of subschemes in $\mathbb{P}(\wedge^2 W \oplus \wedge^2 W)$ defined by

$$\Sigma_t = \{[q, p] \in \mathbb{P}(\wedge^2 W \oplus \wedge^2 W) \mid p \wedge p = tq \wedge q + 2q \wedge p = 0\}.$$

In particular, $(\Sigma_0)_{red} = \Sigma$ holds by the previous lemma.

Lemma 4.6. *For $t \in \mathbb{C} \setminus 0$, Σ_t is projectively equivalent to the join of two Grassmannians*

$$G(2, W) \subset \mathbb{P}(\wedge^2 W \oplus \{0\}), \quad G(2, W) \subset \mathbb{P}(\{0\} \oplus \wedge^2 W).$$

In particular, Σ_t is reduced for $t \neq 0$. Furthermore, the family $\{(\Sigma_t)_{red}\}_{t \in \mathbb{C}}$ is flat.

Proof. Assume $t \neq 0$. Since

$$(tq + p) \wedge (tq + p) = t^2 q \wedge q + 2tq \wedge p + p \wedge p = t(tq \wedge q + 2q \wedge p) + p \wedge p,$$

we have

$$\{[q, p] \mid p \wedge p = tq \wedge q + 2q \wedge p = 0\} = \{[q, p] \mid p \wedge p = (tq + p) \wedge (tq + p) = 0\} \quad (59)$$

in $\mathbb{P}(\wedge^2 W \oplus \wedge^2 W)$. Since the right hand side of (59) is projectively equivalent to

$$\{[q, p] \mid p \wedge p = q \wedge q = 0\} \subset \mathbb{P}(\wedge^2 W \oplus \wedge^2 W),$$

which is nothing but the join of the two $G(2, W)$'s in the statement of this lemma.

To see the flatness, it suffices to show that $(\Sigma_t)_{red}$ is normal for any $t \in \mathbb{C}$ by [Har77, Section 3, Theorem 9.11]. For $t \neq 0$, the normality of $(\Sigma_t)_{red} = \Sigma_t$ follows from that of $G(2, W)$. For $(\Sigma_0)_{red} = \Sigma$, we note that the natural map

$$\mathrm{Sym}^k(\wedge^2 W \oplus \wedge^2 W)^* = \mathrm{Sym}^k H^0(G(2, W), \mathcal{K}^* \oplus \mathcal{O}(1)) \rightarrow H^0(G(2, W), \mathrm{Sym}^k(\mathcal{K}^* \oplus \mathcal{O}(1)))$$

is surjective for any $k \geq 0$. Thus $\Sigma \subset \mathbb{P}(\wedge^2 W \oplus \wedge^2 W)$ is projectively normal, and hence normal. Thus we have the flatness. \square

The Calabi–Yau 3-fold which is the intersection of two Grassmannians $G(2, 5)$ with general positions in \mathbb{P}^9 is discussed in [Kan12, Miu13, Kap13]. By Lemma 4.6, we get the following proposition concerning such Calabi–Yau 3-fold.

Proposition 4.7. *Let $Z \subset G(2, W) = G(2, 5)$ be the zero locus of a general section of $\wedge^2 \mathcal{Q}(1)$. Then Z is a flat degeneration of Calabi–Yau 3-folds of type $G(2, W) \cap G(2, W) \subset \mathbb{P}(\wedge^2 W)$.*

Proof. Let s be a general section of $\wedge^2 \mathcal{Q}(1)$ on $G(2, W)$. Since $H^0(\wedge^2 \mathcal{Q}) \otimes H^0(\mathcal{O}(1)) \rightarrow H^0(\wedge^2 \mathcal{Q}(1))$ is surjective, we can take a lift $\bar{s} \in H^0(\wedge^2 \mathcal{Q}) \otimes H^0(\mathcal{O}(1))$ of s . Hence we can apply Proposition 4.1, and Z is isomorphic to a linear section of Σ by $P_{\bar{s}}$. Since $\Sigma_t = (\Sigma_t)_{red}$ degenerates to $(\Sigma_0)_{red} = \Sigma$, $\Sigma_t \cap P_{\bar{s}}$ degenerates to $\Sigma \cap P_{\bar{s}} \simeq Z$. By Lemma 4.6, Σ_t is projectively equivalent to the join of two Grassmannians in $\mathbb{P}(\wedge^2 W \oplus \wedge^2 W)$. Since $\mathbb{P}(\wedge^2 W) \simeq P_{\bar{s}}$ is general in $\mathbb{P}(\wedge^2 W \oplus \wedge^2 W)$, each $\Sigma_t \cap P_{\bar{s}}$ is a variety of type $G(2, W) \cap G(2, W) \subset \mathbb{P}(\wedge^2 W)$ if t is sufficiently close to 0, and we obtain this proposition. \square

4.3. No. 10. Manivel [Man15, Theorem 3.1] proved that the zero locus of a general section of $\mathcal{Q}(1)$ on $G(2, n)$ is projectively equivalent to a general linear section of $G(2, n+1)$ of codimension n . We can regard Proposition 4.1 as a generalization of [Man15, Theorem 3.1] as follows.

Consider the diagram

$$\begin{array}{ccc} & G_{G(2, W)}(2, \mathcal{S} \oplus \mathcal{O}_{G(2, W)}) & \\ \mu \swarrow & & \searrow \pi \\ G(2, W \oplus \mathbb{C}) & & G(2, W), \end{array}$$

where π is the Grassmannian bundle and μ is the morphism induced by $\mathcal{S} \oplus \mathcal{O}_{G(2,W)} \subset (W \oplus \mathbb{C}) \otimes \mathcal{O}_{G(2,W)}$. Since

$$\begin{aligned} G_{G(2,W)}(2, \mathcal{S} \oplus \mathcal{O}_{G(2,W)}) &= \mathbb{P}(\mathcal{S}^* \oplus \mathcal{O}_{G(2,W)}) \\ &\simeq \mathbb{P}(\mathcal{S}(1) \oplus \mathcal{O}_{G(2,W)}) \simeq \mathbb{P}(\mathcal{S} \oplus \mathcal{O}_{G(2,W)}(-1)), \end{aligned}$$

this diagram in nothing but the diagram (51) for $\mathcal{E} = \mathcal{Q}$ and

$$\Sigma = G(2, W \oplus \mathbb{C}) \subset \mathbb{P}(\wedge^2(W \oplus \mathbb{C})) = \mathbb{P}(W \oplus \wedge^2 W).$$

Hence [Man15, Theorem 3.1] follows from Proposition 4.1.

5. ALTERNATIVE DESCRIPTION: NO. 16 AND 21

In this section, we show the following proposition, which states that Calabi–Yau 3-folds in No. 16 (resp. No. 21) are deformation equivalent to general linear sections of $G(2, 7)$ of codimension 7 (resp. general linear sections of $G(3, 6)$ of codimension 6).

Proposition 5.1. *Let $n = \dim W$ and let $Z \subset G(k, W)$ be the zero locus of a general section of $\mathcal{S}^*(1) \oplus \wedge^{n-k-1} \mathcal{Q}$. Then Z is a flat degeneration of general complete intersections $Z_{\mathcal{O}(1)^{\oplus n}} \subset G(k, W)$.*

Proof. We note that $\wedge^{n-k-1} \mathcal{Q}$ is isomorphic to $\mathcal{Q}^*(1)$. By the exact sequence

$$0 \rightarrow \mathcal{Q}^*(1) \rightarrow W^*(1) \rightarrow \mathcal{S}^*(1) \rightarrow 0, \quad (60)$$

we have an exact sequence of global sections

$$0 \rightarrow H^0(\mathcal{Q}^*(1)) \rightarrow H^0(W^*(1)) \xrightarrow{\varpi} H^0(\mathcal{S}^*(1)) \rightarrow 0.$$

Choose and fix general $(s, q) \in H^0(\mathcal{S}^*(1)) \oplus H^0(\mathcal{Q}^*(1))$. Since s is general, there exists a general section $\bar{s} \in H^0(W^*(1))$ such that $\varpi(\bar{s}) = s$. Let $X \subset G(k, W)$ be the zero locus of $s \in H^0(\mathcal{S}^*(1))$. On X , we have an exact sequence

$$0 \rightarrow H^0(X, \mathcal{Q}^*(1)|_X) \rightarrow H^0(X, W^*(1)|_X) \xrightarrow{\varpi} H^0(X, \mathcal{S}^*(1)|_X).$$

Since $\varpi(\bar{s}|_X) = s|_X = 0$, $\bar{s}|_X \in H^0(X, W^*(1)|_X)$ is contained in $H^0(X, \mathcal{Q}^*(1)|_X)$. By the exact sequence (60), the zero locus of $\bar{s} \in H^0(W^*(1))$ in $G(k, W)$ coincides with the zero locus of $\bar{s}|_X \in H^0(X, \mathcal{Q}^*(1)|_X)$ in X .

Let $Z_t \subset G(k, W)$ be the zero locus of $q + t\bar{s} \in H^0(W^*(1))$ for $t \neq 0$. Since $\varpi(q + t\bar{s}) = ts$, Z_t is contained in X . As a subscheme of X , Z_t is the zero locus of $q + t\bar{s}|_X \in H^0(X, \mathcal{Q}^*(1)|_X)$. Since Z is the zero locus of $q|_X \in H^0(X, \mathcal{Q}^*(1)|_X)$ as a subscheme of X , Z_t degenerates to Z .

The Koszul complex induced by $(s, q) \in H^0(\mathcal{S}^*(1)) \oplus H^0(\mathcal{Q}^*(1))$ (resp. $q + t\bar{s} \in H^0(W^*(1))$) gives a locally free resolution of \mathcal{O}_Z (resp. \mathcal{O}_{Z_t}) on $G(k, W)$. Hence Z and Z_t have the same Hilbert polynomial by the exact sequence (60). Thus this degeneration is flat. \square

Remark 5.2. By Proposition 5.1, we see the G_2 -Grassmann Calabi–Yau 3-fold X in [IMOUa, IMOUb] is a specialization of linear section Calabi–Yau 3-folds $Z_{\mathcal{O}(1)^{\oplus 7}} \subset G(2, 7)$.

6. ALTERNATIVE DESCRIPTION: NO. 17 AND 18

In this section, we give an alternative description of $Z_{\wedge^5 \mathcal{Q}}$ in $G(2, 8)$.

We recall the definition and some facts of the space of determinantal nets of conics. See [EPS87], [Tjø97] for the detail.

Let V, E and F be \mathbb{C} -vector spaces of dimension 3, 3 and 2, respectively. Consider the group action $GL(E) \times GL(F)$ on $\text{Hom}(F, E \otimes V)$ defined by

$$(g, h)\alpha = (g \otimes \text{id}_V) \circ \alpha \circ h^{-1}$$

for $g \in GL(E), h \in GL(F)$, and $\alpha \in \text{Hom}(F, E \otimes V)$. Since the normal subgroup $\Gamma = \{t(\text{id}_E, \text{id}_F) \mid t \in \mathbb{C}^*\}$ acts on $\text{Hom}(F, E \otimes V)$ trivially, the group $G = GL(E) \times GL(F)/\Gamma$ acts on $\text{Hom}(F, E \otimes V)$.

A point of $\text{Hom}(F, E \otimes V)$ is called *stable* (resp. *semistable*) if so is the corresponding point of $\mathbb{P}(\text{Hom}(F, E \otimes V))$ in the sense of [MF82] for the induced action of $G \cap SL(\text{Hom}(F, E \otimes V))$. In fact, it is shown in [EPS87] that $\text{Hom}(F, E \otimes V)^s = \text{Hom}(F, E \otimes V)^{ss}$ holds. By geometric invariant theory, we have a projective geometric quotient

$$\mathbf{N} = \text{Hom}(F, E \otimes V)^{ss}/G,$$

which is called the space of determinantal nets of conics. As seen in [EPS87], \mathbf{N} is a smooth projective variety of dimension 6.

On \mathbf{N} , there exist vector bundles \mathcal{E}, \mathcal{F} induced by trivial bundles

$$E \otimes \frac{\det F}{\det E} \times \text{Hom}(F, E \otimes V)^{ss}, \quad F \otimes \frac{\det F}{\det E} \times \text{Hom}(F, E \otimes V)^{ss}$$

on $\text{Hom}(F, E \otimes V)^{ss}$, respectively. The tautological map on $\text{Hom}(F, E \otimes V)^{ss}$ induces a homomorphism

$$\mathcal{A} : \mathcal{F} \rightarrow \mathcal{E} \otimes V.$$

The Picard group of \mathbf{N} is generated by $\det \mathcal{E}^* \simeq \det \mathcal{F}^*$.

In [Kuz15, Theorem 4.10], Kuznetsov proved that $Z_{\wedge^3 \mathcal{Q}}$ in $G(3, 8)$ is isomorphic to the blowup of \mathbb{P}^5 along the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. By a similar strategy, we can show the following proposition.

Proposition 6.1. *The zero locus $Z_{\wedge^5 \mathcal{Q}}$ in $G(2, 8)$ is isomorphic to \mathbf{N} . In particular, a Calabi–Yau 3-fold in No. 17 is isomorphic to a linear section of \mathbf{N} of codimension 3, and a Calabi–Yau 3-fold in No. 18 is isomorphic to the zero locus in \mathbf{N} of a section of $\text{Sym}^2 \mathcal{F}^*$.*

Proof. Step 1. In this step, we show that there exists a morphism $\iota : \mathbf{N} \rightarrow G(2, 8)$.

Let $A = (L_{ij})$ be a 3×2 matrix representing $\alpha \in \text{Hom}(F, E \otimes V)$ with $L_{ij} \in V$. Let $E_\alpha \subset \text{Sym}^2 V$ be the subspace spanned by 2×2 minors of A . We note that E_α does not depend on the choice of A . By [EPS87], α is stable if and only if α is semistable if and only if $\dim E_\alpha = 3$. Hence \mathcal{E} is embedded in $\text{Sym}^2 V \otimes \mathcal{O}_{\mathbf{N}}$ as a subbundle, and we have a complex

$$0 \rightarrow \mathcal{F} \xrightarrow{\mathcal{A}} \mathcal{E} \otimes V \xrightarrow{m} \text{Sym}^3 V \otimes \mathcal{O}_{\mathbf{N}} \quad (61)$$

on \mathbf{N} , where m is the multiplication map. We note that $\mathcal{E} \hookrightarrow \text{Sym}^2 V \otimes \mathcal{O}_{\mathbf{N}}$ induces an embedding $\mathbf{N} \hookrightarrow G(3, \text{Sym}^2 V)$ as in [Tjø97, Section 3].

Let W be the kernel of the multiplication map

$$\text{Sym}^2 V \otimes V \twoheadrightarrow \text{Sym}^3 V.$$

Since $\dim \text{Sym}^2 V \otimes V = 18$ and $\dim \text{Sym}^3 V = 10$, we have $\dim W = 8$. By the complex (61), \mathcal{A} factors through W , i.e. we have an injection $\mathcal{A} : \mathcal{F} \rightarrow W \otimes \mathcal{O}_{\mathbf{N}}$.

For $\alpha \in \text{Hom}(F, E \otimes V)^{ss}$, we denote by $[\alpha]$ the induced point of \mathbf{N} . We can check that the linear map

$$\mathcal{A}|_{[\alpha]} : \mathcal{F}_{[\alpha]} = F \otimes (\det F / \det E) \rightarrow W$$

is injective (e.g. by taking a matrix $A = (L_{ij})$ representing α and describe $\mathcal{A}|_{[\alpha]}$ by L_{ij}). Hence \mathcal{F} is a subbundle of $W \otimes \mathcal{O}_{\mathbf{N}}$ of rank 2 and defines a morphism $\iota : \mathbf{N} \rightarrow G(2, W)$.

Step 2. In this step, we show that ι is generically injective.

As shown in [EPS87], [Tj97], a general point $[\alpha] \in \mathbf{N}$ is represented by a matrix

$$A = \begin{pmatrix} x & x \\ y & 0 \\ 0 & z \end{pmatrix} \quad (62)$$

for a basis $\{x, y, z\}$ of V . Then we have

$$\mathcal{E}_{[\alpha]} = \langle xy, yz, zx \rangle \subset \text{Sym}^2 V,$$

$$\mathcal{F}_{[\alpha]} = \langle yz \otimes x - xz \otimes y, yz \otimes x - xy \otimes z \rangle \subset W.$$

Hence the image of $\mathcal{F}_{[\alpha]} \otimes V^*$ by the map

$$W \otimes V^* \subset (\text{Sym}^2 V \otimes V) \otimes V^* \rightarrow \text{Sym}^2 V$$

is nothing but $\mathcal{E}_{[\alpha]}$. This means that the point $\mathcal{E}_{[\alpha]} \in G(3, \text{Sym}^2 V)$ is uniquely determined by $\mathcal{F}_{[\alpha]} \in G(2, W)$. Since $\mathbf{N} \hookrightarrow G(3, \text{Sym}^2 V)$ is an embedding, $[\alpha] \in \mathbf{N}$ is uniquely determined by $\mathcal{E}_{[\alpha]} \in G(3, \text{Sym}^2 V)$, hence by $\mathcal{F}_{[\alpha]} \in G(2, W)$. Thus ι is generically injective.

Step 3. By definition, $SL(V)$ acts on $W \subset \text{Sym}^2 V \otimes V$. Similar to [Kuz15, Lemma 4.6], we can check that there exists a unique $SL(V)$ -invariant 5-form $\lambda \in \wedge^5 W$ (up to scalar multiple) by the Littlewood–Richardson rule. As in [Kuz15, Proposition 4.7], we see that the form λ is a general 3-form under the action of $GL(W)$.

Explicitly, λ is written as

$$\lambda = w_{13578} - w_{23578} + w_{12367} - w_{12458} + w_{24567} - w_{13468}, \quad (63)$$

where $w_{ijklm} = w_i \wedge w_j \wedge w_k \wedge w_l \wedge w_m \in \wedge^5 W$ for

$$\begin{aligned} w_1 &= yz \otimes x - xz \otimes y, & w_2 &= yz \otimes x - xy \otimes z, \\ w_3 &= xy \otimes x - x^2 \otimes y, & w_4 &= y^2 \otimes x - xy \otimes y, \\ w_5 &= xz \otimes x - x^2 \otimes z, & w_6 &= z^2 \otimes x - xz \otimes z, \\ w_7 &= yz \otimes y - y^2 \otimes z, & w_8 &= yz \otimes z - z^2 \otimes y. \end{aligned}$$

Step 4. In this step, we finish the proof by showing that the morphism $\iota : \mathbf{N} \rightarrow G(2, W)$ in Step 1 is an embedding and the image is the zero locus Z of $\lambda \in H^0(\wedge^5 \mathcal{Q}) = \wedge^5 W$.

First, we show that $\iota(\mathbf{N})$ is contained in Z . Since λ is $SL(V)$ -invariant, it is enough to show that $\iota([\alpha])$ is contained in Z for the point $[\alpha] \in \mathbf{N}$ represented by the matrix (62).

Since $\iota^* \mathcal{S} = \mathcal{F}$, it holds that

$$\mathcal{S}_{\iota([\alpha])} = \mathcal{F}_{[\alpha]} = \langle yz \otimes x - xz \otimes y, yz \otimes x - xy \otimes z \rangle = \langle w_1, w_2 \rangle.$$

For each term w_{ijklm} in (63), at least one of 1 or 2 appears in $\{i, j, k, l, m\}$. Hence $\lambda \in H^0(\wedge^5 \mathcal{Q})$ vanishes at $\iota([\alpha])$, i.e. $\iota([\alpha]) \in Z$ and we have $\iota(\mathbf{N}) \subset Z$.

Since $\lambda \in \wedge^5 W = H^0(\wedge^5 \mathcal{Q})$ is general, Z is smooth of dimension 6. We can check $h^0(\mathcal{O}_Z) = 1$ and hence Z is irreducible. Since ι is generically injective by Step 2, $\dim \iota(\mathbf{N}) = \dim \mathbf{N} = 6 = \dim Z$. Hence $\iota(\mathbf{N}) = Z$ holds. Since $\iota : \mathbf{N} \rightarrow Z$ is birational, $\rho(\mathbf{N}) = 1$, and Z is smooth (in particular, normal), $\iota : \mathbf{N} \rightarrow Z$ must be an isomorphism by Zariski's Main Theorem and this proposition is proved. \square

7. ALTERNATIVE DESCRIPTION: THE REST CASES

In this section, we see the rest of descriptions in Table 1 briefly.

No. 9 : As in the proof of [Kuz15, Proposition 2.1], $Z_{\text{Sym}^2 \mathcal{S}^*} \subset G(2, 6)$ can be identified with the flag variety $F(1, 3; \mathbb{C}^4) \subset \mathbb{P}^3 \times \mathbb{P}^3$. Under the identification, $\mathcal{O}(1)|_{Z_{\text{Sym}^2 \mathcal{S}^*}}$ and $\mathcal{S}|_{Z_{\text{Sym}^2 \mathcal{S}^*}}$ correspond to

$$\mathcal{O}(1, 1) := \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1, 1)|_{F(1, 3; \mathbb{C}^4)} \quad \text{and} \quad (\mathcal{S}_3/\mathcal{S}_1) \otimes \mathcal{S}_1 \quad (64)$$

on $F(1, 3; \mathbb{C}^4)$ respectively, where $\mathcal{S}_1 \subset \mathcal{S}_3 \subset \mathcal{O}_{F(1, 3; \mathbb{C}^4)}^{\oplus 4}$ are the universal subbundles of rank 1 and 3. Hence $\mathcal{S}^*(1)|_{Z_{\text{Sym}^2 \mathcal{S}^*}}$ corresponds to $(\mathcal{S}_3/\mathcal{S}_1)^* \otimes \mathcal{S}_1^* \otimes \mathcal{O}(1, 1) = (\mathcal{S}_3/\mathcal{S}_1)^* \otimes \mathcal{O}(2, 1)$. Thus the Calabi–Yau 3-fold $Z_{\text{Sym}^2 \mathcal{S}^* \oplus \mathcal{S}^*(1)} \subset G(2, 6)$ is isomorphic to the zero locus of a section of $(\mathcal{S}_3/\mathcal{S}_1)^* \otimes \mathcal{O}(2, 1)$ on $F(1, 3; \mathbb{C}^4)$.

No. 11 : $Z_{\wedge^3 \mathcal{Q}} \subset G(2, 6)$ is a 4-dimensional Del Pezzo manifold with Picard number two. By the classification of Del Pezzo manifolds (see [IP99]), it is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$.

No. 15 and 16 : It is known that $Z_{\wedge^4 \mathcal{Q}} \subset G(2, 7)$ is isomorphic to a rational homogeneous space G_2/P_1 by [MRT], (see [CCGK16, Section 16]).

No. 22 : We see that $Z_{\text{Sym}^2 \mathcal{S}^*} \subset G(3, 7)$ is an orthogonal Grassmannian $OG(3, 7) \simeq OG(4, 8)$. By the triality of $SO(8)$, it is also isomorphic to $OG(1, 8)$, a quadric hypersurface $Q^6 \subset \mathbb{P}^7$, which is regarded as the spinor embedding of $OG(3, 7)$. Since $\mathcal{O}_{G(3, 7)}(1)|_{Z_{\text{Sym}^2 \mathcal{S}^*}} = \mathcal{O}_{Q^6}(2)$ under the above isomorphism, $Z_{\text{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}} \subset G(3, 7)$ is nothing but a complete intersection of four quadric hypersurfaces in \mathbb{P}^7 .

No. 26 : We recall a result by Reid in [Rei72].

Let W be a vector space of dimension $2k + 2$ and let $\mathbb{C}^2 \hookrightarrow \text{Sym}^2 W^*$ be a general pencil of symmetric-forms. Let $X \subset G(k, W)$ be the zero locus of the section of $(\text{Sym}^2 \mathcal{S}^*)^{\oplus 2}$ corresponding to this pencil.

Let $l \subset \mathbb{P}(\text{Sym}^2 W^*)$ be the line corresponding to this pencil and let $D \subset \mathbb{P}(\text{Sym}^2 W^*)$ be the discriminant hypersurface corresponding to degenerate symmetric-forms. Since the pencil is general, the line l intersects with D transversally and $l \cap D$ consists of $2k + 2$ points. Let $C \rightarrow l$ be the hyperelliptic curve ramified over $l \cap D$. Reid proved the following theorem.

Theorem 7.1 ([Rei72, Theorem 4.8]). *X is isomorphic as a variety to the Jacobian $J(C)$.*

We note that $\rho(X) = 1$. In fact, C is general in the space of hyperelliptic curve of genus k since the pencil is general. By [Mor76, Theorem 6.5], the endomorphism ring $\text{End}(J(C))$ is isomorphic to \mathbb{Z} . Since the Néron-Severi group $NS(J(C))$ is naturally embedded into $\text{Hom}(J(C), \widehat{J(C)})$ for the dual abelian variety $\widehat{J(C)}$ (cf. [Mum70]) and $\widehat{J(C)} \simeq J(C)$, we have $NS(J(C)) \simeq \mathbb{Z}$.

No. 30 : Similarly to No. 22, $Z_{\mathrm{Sym}^2 \mathcal{S}^*} \subset G(4, 8)$ is a disjoint union of two quadrics $Q^6 \simeq OG(4, 8)$ and the restriction of $\mathcal{O}_{G(4,8)}(1)$ on each component Q^6 coincides with $\mathcal{O}_{Q^6}(2)$. Hence $Z_{\mathrm{Sym}^2 \mathcal{S}^* \oplus \mathcal{O}(1)^{\oplus 3}} \subset G(4, 8)$ is a disjoint union of two $(\mathbb{P}^7)_{24}$.

No. 31 and 33 : $Z_{(\wedge^2 \mathcal{S}^*)^{\oplus 2}} \subset G(k, 2k)$ is isomorphic to $\prod^k \mathbb{P}^1$ by [Kuz15, Theorem 3.1].

No. 32 : As in [Küc95, Example 4.1], $Z_{\mathrm{Sym}^2 \mathcal{S}^* \oplus \wedge^2 \mathcal{S}^*}$ is of index at least two. We can also compute $h^{1,1} \geq 4$ for $Z_{\mathrm{Sym}^2 \mathcal{S}^* \oplus \wedge^2 \mathcal{S}^*}$. Hence $Z_{\mathrm{Sym}^2 \mathcal{S}^* \oplus \wedge^2 \mathcal{S}^*}$ is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by the classification of Fano 4-folds of index two with Picard number at least two (see [IP99]).

REFERENCES

- [Cas15] C. Casagrande, *Rank 2 quasiparabolic vector bundles on \mathbb{P}^1 and the variety of linear subspaces contained in two odd-dimensional quadrics*, Math. Z. **280** (2015), no. 3-4, 981–988. MR 3369361
- [CCGK16] Tom Coates, Alessio Corti, Sergey Galkin, and Alexander Kasprzyk, *Quantum periods for 3-dimensional Fano manifolds*, Geom. Topol. **20** (2016), no. 1, 103–256. MR 3470714
- [CK99] David A. Cox and Sheldon Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, 1999. MR MR1677117 (2000d:14048)
- [CR13] Izzet Coskun and Colleen Robles, *Flexibility of Schubert classes*, Differential Geom. Appl. **31** (2013), no. 6, 759–774. MR 3130568
- [EPS87] Geir Ellingsrud, Ragni Piene, and Stein Arild Strømme, *On the variety of nets of quadrics defining twisted cubics*, Space curves (Rocca di Papa, 1985), Lecture Notes in Math., vol. 1266, Springer, Berlin, 1987, pp. 84–96. MR 908709
- [Gal14] Sergey Galkin, *An explicit construction of Miura’s varieties*, 2014, Talk presented at Tokyo University, Graduate School for Mathematical Sciences, Komaba Campus, February 20.
- [GKM] Sergey Galkin, Alexander Kuznetsov, and Michael Mavshev, *An explicit construction of Miura’s varieties*, in prepration.
- [Gro95] Alexander Grothendieck, *Techniques de construction et théorèmes d’existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. Exp. No. 221, 249–276. MR 1611822
- [Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 1977, xvi+496 pp. ISBN: 0-387-90244-9.
- [IMOUa] Atsushi Ito, Makoto Miura, Shinnosuke Okawa, and Kazushi Ueda, *Calabi–Yau complete intersections in homogeneous spaces of G_2* , arXiv:1606.04076 [math.AG].
- [IMOUb] ———, *The class of the affine line is a zero divisor in the Grothendieck ring: via G_2 -Grassmannians*, arXiv:1606.04210 [math.AG].
- [IP99] V. A. Iskovskikh and Yu. G. Prokhorov, *Fano varieties*, Algebraic geometry, V, Encyclopaedia Math. Sci., vol. 47, Springer, Berlin, 1999, pp. 1–247. MR 1668579
- [Kan12] Atsushi Kanazawa, *Pfaffian Calabi–Yau threefolds and mirror symmetry*, Commun. Number Theory Phys. **6** (2012), no. 3, 661–696. MR 3021322
- [Kap13] Michał Kapustka, *Mirror symmetry for Pfaffian Calabi–Yau 3-folds via conifold transitions*, arXiv:1310.2304v1 [math.AG], 2013.
- [Küc95] Oliver Küchle, *On Fano 4-fold of index 1 and homogeneous vector bundles over Grassmannians*, Math. Z. **218** (1995), no. 4, 563–575.
- [Kuz15] A. G. Kuznetsov, *On Küchle varieties with Picard number greater than 1*, Izv. Math. **79** (2015), no. 4, 698–709.
- [Kuz16] Alexander Kuznetsov, *Küchle fivefolds of type c5*, arXiv:1603.03161 [math.AG].
- [LM04] Joseph M. Landsberg and Laurent Manivel, *Representation theory and projective geometry*, Algebraic transformation groups and algebraic varieties, Encyclopaedia Math. Sci., vol. 132, Springer, Berlin, 2004, pp. 71–122. MR 2090671

- [Man15] Laurent Manivel, *On Fano manifolds of Picard number one*, Math. Z. **281** (2015), no. 3-4, 1129–1135. MR 3421656
- [MF82] David Mumford and John Fogarty, *Geometric invariant theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 34, Springer-Verlag, Berlin, 1982. MR 719371
- [Miu13] Makoto Miura, *Hibi toric varieties and mirror symmetry*, Ph.D. thesis, The University of Tokyo, 2013.
- [Mor76] Shigefumi Mori, *The endomorphism rings of some Abelian varieties*, Japan. J. Math. (N.S.) **2** (1976), no. 1, 109–130. MR 0453754
- [MRT] Shigeru Mukai, Miles Reid, and Hiromichi Takagi, *Classification of indecomposable Gorenstein Fano 3-folds*, 44pages, unpublished manuscript (unknown date).
- [Muk92] Shigeru Mukai, *Polarized K3 surfaces of genus 18 and 20*, Complex projective geometry (Trieste, 1989/Bergen, 1989), London Math. Soc. Lecture Note Ser., vol. 179, Cambridge Univ. Press, Cambridge, 1992, pp. 264–276. MR 1201388 (94a:14039)
- [Mum70] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970. MR 0282985
- [Nit05] Nitin Nitsure, *Construction of Hilbert and Quot schemes*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 105–137. MR 2223407
- [Rei72] Miles Reid, *The complete intersection of two or more quadrics*, Ph.D. thesis, Cambridge, June 1972, 84pp.
- [Tjø97] Erik N. Tjøtta, *Rational curves on the space of determinantal nets of conics*, Ph.D. thesis, University of Bergen, 1997, arXiv:9802037 [math.AG].

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